Lecture Note

Applied Mathematics for Business and Economics

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Lecture Note

Applied Mathematics for Business and Economics

Contents

Page

Chapter 1 Functions

1 Definition of a Function (of one variable) 1
 1.1 Definition 1
 1.2 Domain of a Function 1
 1.3 Composition of Functions 2

2 The Graph of a Function 3

3 Linear Functions 5
 3.1 The Slope of a Line 5
 3.2 Horizontal and Vertical Lines 6
 3.3 The Slope-Intercept Form 6
 3.4 The Point-Slope Form 6

4 Functional Models 8
 4.1 A Profit Function 8
 4.2 Functions Involving Multiple Formulas 8
 4.3 Break-Even Analysis 9
 4.4 Market Equilibrium 11

Chapter Exercises 12

Chapter 2 Differentiation: Basic Concepts

1 The Derivative
 Definition 19

2 Techniques of Differentiation 20
 2.1 The Power Rule 20
 2.2 The Derivative of a constant 21
 2.3 The Constant Multiple Rule 21
 2.4 The Sum Rule 21
 2.5 The Product Rule 21
 2.6 The Derivative of a Quotient 21

3 The Derivative as a Rate of change 22
 3.1 Average and Instantaneous Rate of Change 22
 3.2 Percentage Rate of Change 23

4 Approximation by Differentials; Marginal Analysis 23
 4.1 Approximation of Percentage change 24
 4.2 Marginal Analysis in Economics 25
 4.3 Differentials 27

5 The Chain Rule 27

6 Higher-Order Derivatives 29
 6.1 The Second Derivative 29
 6.2 The n^{th} Derivative 30

7 Concavity and the Second Derivative Test......30

8 Applications to Business and Economics......34
8.1 Elasticity of Demand......34
8.2 Levels of Elasicity of Demand......36
8.3 Elasticity and the Total Revenue......36
Chapter Exercises......38

Chapter 3 Functions of Two Variables
1 Functions of Two Variables......49

2 Partial Derivatives......50
2.1 Computation of Partial Derivatives......50
2.2 Second-Order Partial Derivatives......52

3 The Chain Rule; Approximation by the Total Differential......53
3.1 Chain Rule for Partial Derivatives......53
3.2 The Total differential......55
3.3 Approximation of Percentage Change......56

4 Relative Maxima and Minima......56

5 Lagrange Multipliers......59
5.1 Contrained Optimization Problems......59
5.2 The Lagrange Multiplier......61

Chapter Exercises......62

Chapter 4 Linear Programming (LP)
1 System of Linear Inequalities in Two Variables......72
1.1 Graphing a Linear Inequality in Two Variables......72
1.2 Solving Systems of Linear Inequalities......73

2 Geometric Linear Programming......74

Chapter Exercises......77

Bibliography......81

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Chapter 1

Functions

1 Definition of a Function

1.1 Definition

Let D and R be two sets of real numbers. A function f is a rule that matches each number x in D with exactly one and only one number y or $f(x)$ in R. D is called the domain of f and R is called the range of f. The letter x is sometimes referred to as **independent variable** and y **dependent variable.**

Examples 1:

Let $f(x) = x^3 - 2x^2 + 3x + 100$. Find $f(2)$.

Solution:

$$f(2) = 2^3 - 2\times 2^2 + 3\times 2 + 100 = 106$$

Examples 2

A real estate broker charges a commission of 6% on Sales valued up to \$300,000. For sales valued at more than \$ 300,000, the commission is \$ 6,000 plus 4% of the sales price.

a. Represent the commission earned as a function R.
b. Find R (200,000).
c. Find R (500,000).

Solution

a. $R(x) = \begin{cases} 0.06x & \text{for } 0 \le x \le 300,000 \\ 0.04x + 6000 & \text{for } x > 300,000 \end{cases}$

b. Use $R(x) = 0.06x$ since $200,000 < 300,000$

$$R(200,000) = 0.06\times 200,000 = \$12,000$$

c. Use $R(x) = 0.04x + 6000$ since $500,000 > 300,000$

$$R(500,000) = 0.04\times 500,000 + 6000 = \$26,000$$

1.2 Domain of a Function

The set of values of the independent variables for which a function can be evaluated is called the **domain of the function**.

$$D = \{x \in \mathbb{R} / \exists y \in \mathbb{R}, y = f(x)\}$$

Example 3

Find the domain of each of the following functions:

$$\textbf{a. } f(x) = \frac{1}{x-3}, \quad \textbf{b. } g(x) = \sqrt{x-2}$$

Solution

a. Since division by any real number except zero is possible, the only value of x for which $f(x) = \dfrac{1}{x-3}$ cannot be evaluated is $x = 3$, the value that makes the denominator of f equal to zero, or $D = \mathbb{R} - \{3\}$.

b. Since negative numbers do not have real square roots, the only values of x for which $g(x)=\sqrt{x-2}$ can be evaluated are those for which $x-2$ is nonnegative, that is, for which $x-2\geq 0$ or $x\geq 2$ or $D=[2,+\infty)$.

1.3 Composition of Functions

The composite function $g\left[h(x)\right]$ is the function formed from the two functions $g(u)$ and $h(x)$ by substituting $h(x)$ for u in the formula for $g(u)$.

Example 4

Find the composite function $g\left[h(x)\right]$ if $g(u)=u^2+3u+1\, and\, h(x)=x+1$.

Solution

Replace u by $x+1$ in the formula for g to get.

$$g\left[h(x)\right]=(x+1)^2+3(x+1)+1=x^2+5x+5$$

Example 5

An environmental study of a certain community suggests that the average daily level of carbon monoxide in the air will be $C(p)=0.5p+1$ parts per million when the population is p thousand. It is estimated that t years from now the population of the community will be $P(t)=10+0.1t^2$ thousand.

a. Express the level of carbon monoxide in the air as a function of time.
b. When will the carbon monoxide level reach 6.8 parts per million?

Solution

a. Since the level of carbon monoxide is related to the variable p by the equation.

$$C(p)=0.5p+1$$

and the variable p is related to the variable t by the equation.

$$P(t)=10+0.1t^2$$

It follows that the composite function

$$C\left[P(t)\right]=C\left(10+0.1t^2\right)=0.5\left(10+0.1t^2\right)+1=6+0.05t^2$$

expresses the level of carbon monoxide in the air as a function of the variable t.

b. Set $C\left[P(t)\right]$ equal to 6.8 and solve for t to get

$$\begin{aligned} 6+0.05t^2 &= 6.8 \\ 0.05t^2 &= 0.8 \\ t^2 &= 16 \\ t &= 4 \end{aligned}$$

That is, 4 years from now the level of carbon monoxide will be 6.8 parts per million.

2 The Graph of a Function

The graph of a function *f* consists of all points (x, y) where *x* is in the domain of *f* and $y = f(x)$.

How to Sketch the Graph of a Function *f* by Plotting Points

1 Choose a representative collection of numbers *x* from the domain of *f* and construct a table of function values $y = f(x)$ for those numbers.

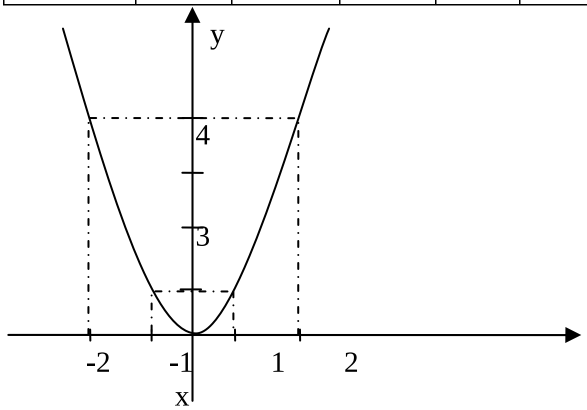
2 Plot the corresponding points (x, y)

3 Connect the plotted points with a smooth curve.

Example 1

Graph the function $y = x^2$. Begin by constructing the table.

x	-2	-1	0	1	2
$y = x^2$	4	1	0	1	4



Example 2 Graph the function

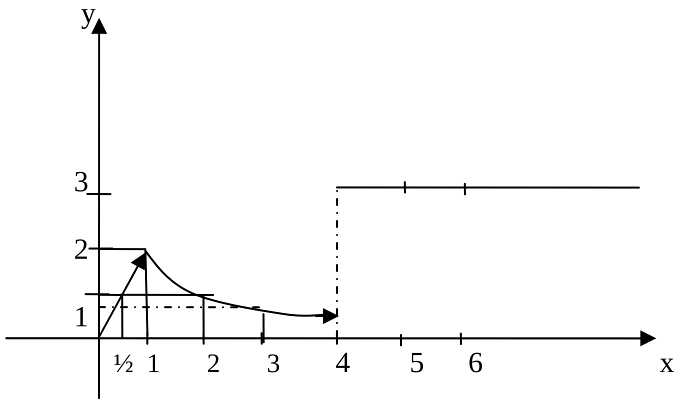
$$f(x) = \begin{cases} 2x, & \text{if } 0 \le x < 1 \\ \dfrac{2}{x}, & \text{if } 1 \le x < 4 \\ 3, & \text{if } x \ge 4 \end{cases}$$

Solution

When making a table of values for this function, remember to use the formula that is appropriate for each particular value of *x*. Using the formula $f(x) = 2x$ when $0 \le x < 1$, the formula $f(x) = 2/x$ when $1 \le x < 4$ and the formula $f(x) = 3$ when $x \ge 4$, you can compile the following table:

x	0	1/2	1	2	3	4	5	6
$f(x)$	0	1	2	1	2/3	3	3	3

Now plot the corresponding point $(x, f(x))$ and draw the graph as in Figure.



Comment

The graph of $y = f(x) = ax^2 + bx + c$ is a **parabola** as long as $a \neq 0$. All parabolas have a **U shape**, and $y = f(x) = ax^2 + bx + c$ opens either up (if $a > 0$) or down (if $a < 0$). The "**Peak**" or "**Valley**" of the parabola is called its **vertex**, and in either case, the *x* coordinate of the vertex is $x = -\dfrac{b}{2a}$.

Note that to get a reasonable sketch of the parabola $y = ax^2 + bx + c$, you need only determine.

1 The location of the vertex
2 Whether the parabola opens up ($a > 0$) or down ($a < 0$)
3 Any intercepts.

Example 3

For the equation $y = x^2 - 6x + 4$

- **a.** Find the Vertex.
- **b.** Find the minimum value for *y*.
- **c.** Find the x-intercepts.
- **d.** Sketch the graph.

Solution

a. We have $a = 1, b = -6,$ and $c = 4$. The vertex occurs at $x = -\dfrac{-6}{2 \times 1} = 3$

Substituting *x* = 3 gives $y = 3^2 - 6 \times 3 + 4 = -5$. The vertex is $(3, -5)$.

b. Since $a = 1 > 0$ and the parabola opens upward, $y = -5$ is the minimum value for *y*.

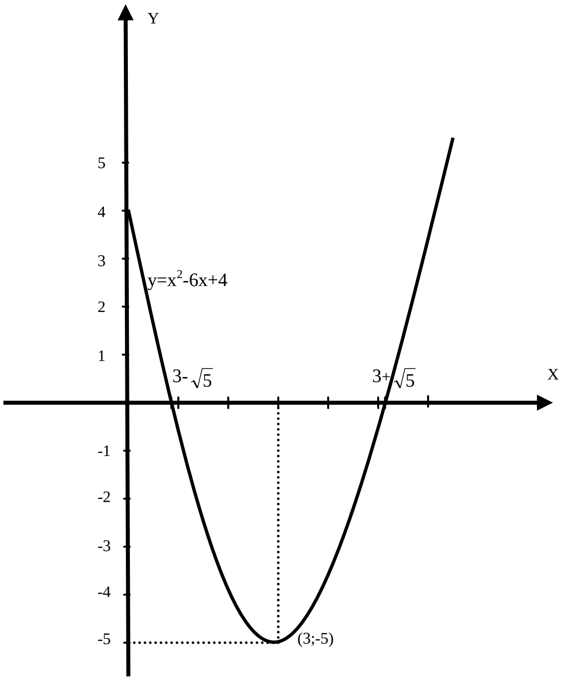
c. The *x*-intercept are found by setting $x^2 - 6x + 4 = 0$ and solving for *x*

$$x = \frac{6 \pm \sqrt{36 - 16}}{2} = 3 \pm \sqrt{5}$$

d. The graph opens upward because $a = 1 > 0$ .The vertex is $(3, -5)$

The axis of symmetry is $x = 3$.

The *x*-intercepts are $x = 3 \pm \sqrt{5}$.



Example 4

A manufacturer can produce radios at a cost of \$10 apiece an estimated that if they are sold for x dollars, consumers will buy approximately $80 - x$ radios each month. Express the manufacturer's monthly profit as a function of the price x, graph this function, and determine the price at which the manufacture's profit will be greatest.

Solution

Begin by stating the desired relationship in words:

Profit = (number of radios sold) (profit per radio)

Now replace the words by algebraic expressions. You know that :

Number of radios sold = 80 – x

and since the radios are produced at a cost of \$10 apiece and sold for x dollars apiece,

It follows that profit per radio $= 80 - x$

Let $P(x)$ denotes the profit and conclude that

$$P(x) = (80 - x)(x - 10) = -x^2 + 90x - 800$$

3 Linear Functions

Linear function is a function that changes at a constant rate with respect to its independent variable. The graph of a linear function is a straight line. The equation of a linear function can be written in the form

$$y = mx + b$$

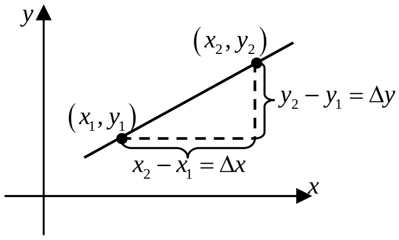
where m and b are constants.

3.1 The Slope of a Line

The **slope** of a line is the amount by which the y coordinate of a point on the line changes when the x coordinate is increased by 1.

The Slope of a Line
The slope of the nonvertical line passing thruough the points (x_1, y_1) and (x_2, y_2) is given by the formula

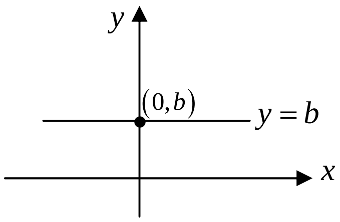
$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



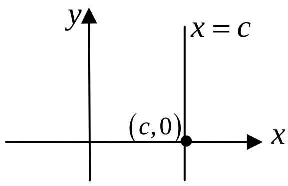
Example 1
Find the slope of the line joining the points $(-2, 5)$ and $(3, -1)$.

3.2 Horizontal and Vertical Lines

The horizontal line has the equation $y = b$, where b is a constant. Its slope is equal to zero. The vertical line has the equation $x = c$, where c is a constant. Its slope is undefined. See the figure.



Horizontal line



Vertical line

3.3 The Slope-Intercept Form

The Slope-Intercept Form of the Equation of a Line
The equation

$$y = mx + b$$

is the equation of the line whose slope is m and whose y intercept is the point $(0, b)$

Example 2
Find the slope and y intercept of the line $3y + 2x = 6$ and draw the graph.

3.4 The Point-Slope Form

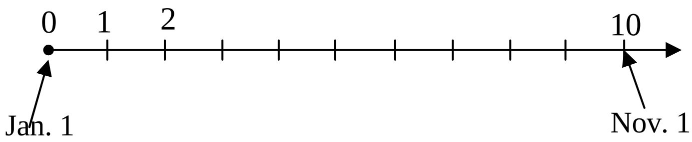
The Point-Slope Form of the Equaiton of a Line
The equation $y - y_0 = m(x - x_0)$ is and equation of the line that passes through the point (x_0, y_0) and that slope equal to *m*.

Example 3
Find an equation of the line that passes through the point $(5,1)$ and whose slope is equal to1/2.

Example 4
Find an equation of the line that passes through the points $(3, -2)$ and $(1, 6)$.

Example 5

Since the beginning of the year, the price of whole-wheat bread at a local discount supermarket has been rising at a constant rate of 2 cents per month. By November 1, the price had reached \$1.06 per loaf. Express the price of the bread as a function of time and determine the price at the beginning of the year.



Solution

Let x: denote the number of months that have elapsed since January 1

y: denote the price of a loaf of bread (in cents).

Since y changes at a constant rate with respect to x, the function relating y to x must be linear and its graph is a straight line. Because the price y increases by 2 each time x increase by 1, the slope of the line must be 2. Then, we have to write the equation of the line with slope 2 and passes throught the point $(10,106)$. By the fomular, we obtain

$$y - y_0 = m(x - x_0)$$
$$y - 106 = 2(x - 10)$$

or

$$y = 2x + 86$$

At the beginning of the year, we have $x = 0$, then $y = 86$. Hence, the price of tbread at the beginning of the year was 86 cents per loaf.

Example 6

The average scores of incoming students at an eastern liberal arts college in the SAT mathematics examination have been declining at a constant rate in recent years. In 1986, the average SAT score was 575, while in 1991 it was 545.

a. Express the average SAT score as a function of time.
b. If the trend continues, what will the average SAT score of incoming students be in 1996?
c. If the trend continues, when will the average SAT score be 527?

(Answer: a. $y = -6x + 575$, b. 515, c. 8)

Example 7

A manufacturer's total cost consists of a fixed overhead of \$ 200 plus production costs of \$ 50 per unit. Express the total cost as a function of the number of units produced and draw the graph.

Solution

Let x denote the number of units produced and $C(x)$ the corresponding total cost. Then, Total cost = (cost per unit) (number of units) + overhead.

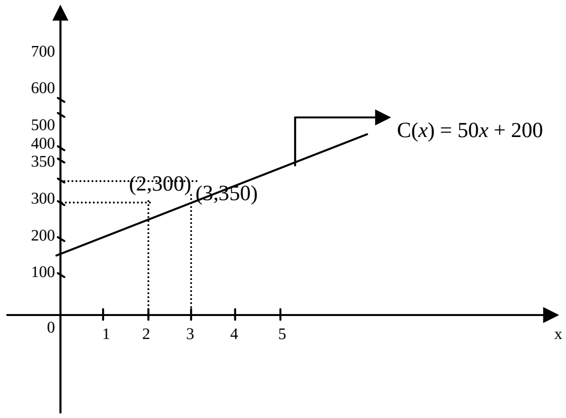
Where Cost per unit = 50

Number of units = x

Overhead = 200

Hence,

$$C(x) = 50x + 200$$



4 Functional Models

4.1 A Profit Function

In the following example, profit is expressed as a function of the price at which a product is sold.

Example 1

A manufacturer can produce radios at a cost of \$ 2 apiece. The radios have been selling for \$ 5 apiece, and at this price, consumers have been buying 4000 radios a month. The manufacturer is planning to raise the price of the radios and estimate that for each \$1 increase in the price, 400 fewer radios will be sold each month. Express the manufacturer's monthly profit as a function of the price at which the radios are sold.

Solution

Begin by stating the desired relationship in words.

Profit = (number of radios sold) (Profit per radio)

Let x denote the price at which the radios will be sold and $P(x)$ the corresponding profit.

Number of radios sold = 4000 – 400 (number of \$ 1 increases) the number of \$ 1 increases in the price is the difference x-5 between the new and old selling prices. Hence,

Number of radios sold = 4000 – 400 (x - 5)

$= 400[10-(x-5)] = 400(15-x)$

Profit per radio = x – 2

$$P(x) = 400(15-x)(x-2)$$

4.2 Functions Involving Multiple Formulas

In the next example, you will need three formulas to define the desired function.

Example 2

During a drought, residents of Marin Country, California, were faced with a severe water shortage. To discourage excessive use of water, the country water district initiated drastic rate increases. The monthly rate for a family of four was \$ 1.22 per 100 cubic feet of water for the first 1,200 cubic feet, \$10 per 100 cubic feet for the next 1200 cubic feet, and $50 per 100 cubic feet there after. Express the monthly water bill for a family of four as a function of the amount of water used.

Solution

Let x denote the number of hundred-cubic-feet units of water used by the family during the month and $C(x)$ the corresponding cost in dollars. If $0 \le x \le 12$, the cost is simply the cost per unit times the number of units used.

$$C(x) = 1.22x$$

If $12 \le x \le 24$ each of the first 12 units cost $1.22, and so the total cost of these 12 units is 1.22(12) = 14.64 dollars. Each of the remaining $x-12$ units costs $10, and hence the total cost of these units is $10(x-12)$ dollars. The cost of all x units is the sum.

$$C(x) = 14.64 + 10(x-12) = 10x - 105.36$$

If $x > 24$, the cost of the first 12 units is 1.22 (12) = 14.64 dollars, the cost of the next 12 units is 10 (12) = 120, and the cost of the remaining $x-24$ units is so $(x-24)$ dollars. The cost of all x units is the sum.

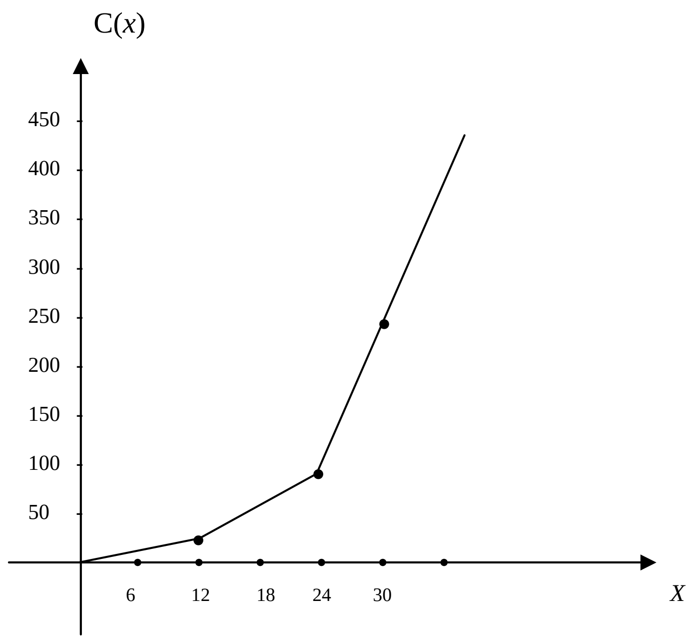
$$C(x) = 14.64 + 120 + 50(x-24) = 50x - 1,065.36$$

Combining these three formulas, you get.

$$C(x) = \begin{cases} 1.22x, & \text{if } 0 \le x \le 12 \\ 10x - 105.36 & \text{if } 12 \le x \le 24 \\ 50x - 1,065.36 & \text{if } x > 24 \end{cases}$$

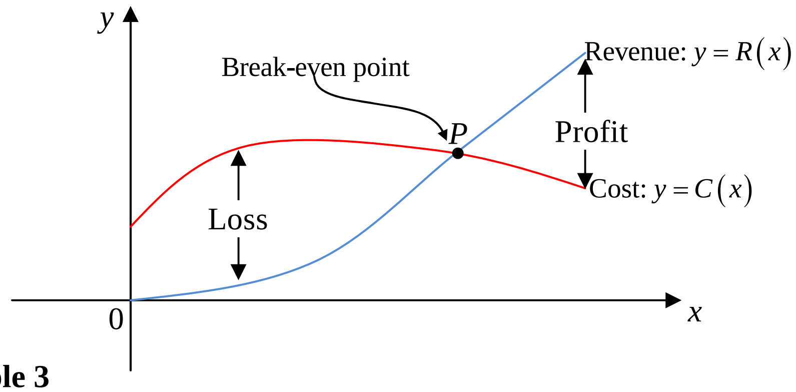
The graph of this function

x	$C(x)$
0	0
12	14.64
24	134.64
30	434.64



4.3 Break-Even Analysis

Intersections of graphs arise in business in the context of **break-even analysis**. In a typical situation, a manufacturer wishes to determine how many units of a certain commodity have to be sold for total revenue to equal total cost. Suppose x denotes the number of units manufactured and sold, and let $C(x)$ and $R(x)$ be the corresponding total cost and total revenue, respectively. A pair of cost and revenue curves is sketched in Figure:



Example 3

The Green-Belt Company determines that the cost of manufacturing men's belts is \$ 2 each plus \$ 300 per day in fixed costs. The company sells the belts for \$ 3 each. What is the break-even point?

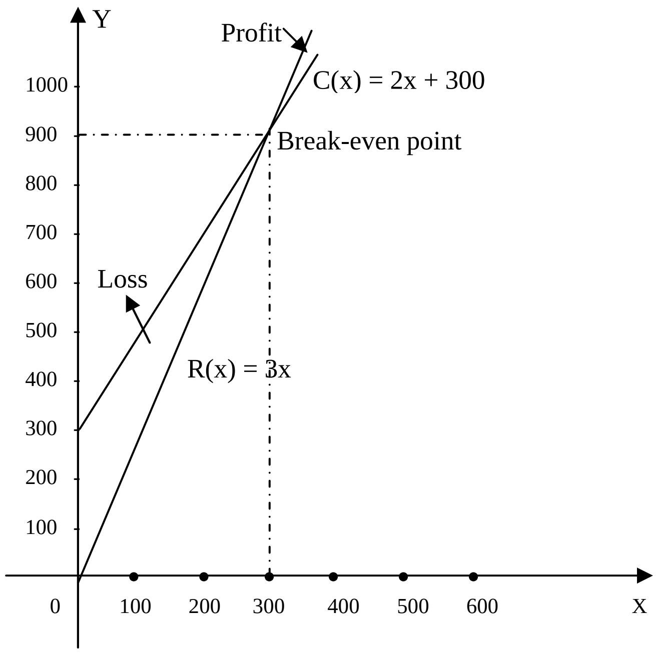
Solution

The break-even point occurs where revenue and cost are equal. By letting x = the number of belts manufactured in a day, and then we obtain revenue funcion $R(x) = 3x$ and cost function $C(x) = 2x + 300$. For $C(x) = R(x)$, we obtain

$$3x = 2x + 300$$

$$x = 300$$

So, 300 belts must be sold each day for the company to break even. The company must sell more than 300 belts each day to make a profit.



Example 4

Suppose that a company has determined that the cost of producing x items is 500 + 140x and that the price it should charge for one item is P = 200 – x

a. Find the cost function.
b. Find the revenue function.
c. Find the profit function.
d. Find the break-even point

Solution

a. The cost function is given: $C(x) = 500 + 140x$

b. The revenue function is found by multiplying the price for one item by the number of items sold.

$$R(x) = P \times x = (200 - x)x = 200x - x^2$$

c. Profit is the difference between revenue and cost

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (200x - x^2) - (500 + 140x) \\ &= -x^2 + 60x - 500 \end{aligned}$$

d. To find the break- event, set the revenue equal to the cost and solve for x

$$R(x) = C(x)$$

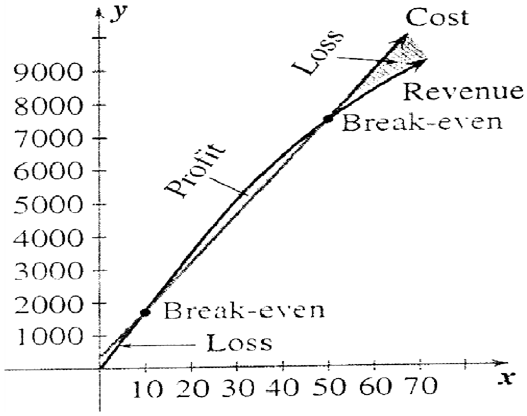
$$200x - x^2 = 500 + 140x$$

$$x^2 - 60x + 500 = 0$$

$$(x - 10)(x - 50) = 0$$

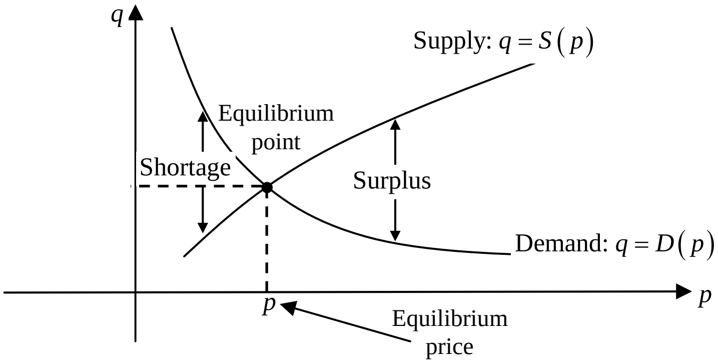
$$x = 10 \text{ or } x = 50$$

This model shows that a profit occurs if the company produces between 10 and 50 items. We will discuss calculus techniques for maximizing profit later.



4.4 Market Equilibrium

An important economic application involving intersections of graphs arises in connection with the **law of supply and demand.** In this context, we think of the market price *p* of a commodity as determining the number of units of the commodity that manufacturers are willing to supply as well as the number of units that consumers are willing to buy. In most cases, manufacturers's supply $S(p)$ increases and consumers'demand $D(p)$ decreases as the market price *p* increases. See the figure.



The point of intersection of the supply and demand curves is called the point of **market equilibrium.** The *p* coordinate of this point (the **equilibrium price**) is the market price at which supply equals demand. We can say another way that the market price is a price at which there will be neither a surplus nor a shortage of the commodity.

Example 5
Find the equilibrium price and the corresponding number of units supplied and demanded if the supply function for a certain commodity is $S(p)=p^2+3p-70$ and the demand function is $D(p)=410-p$.

Solution
Set $S(p)$ equal to $D(p)$ and solve for p to get

$$\begin{aligned} &p^2+3p-70=140-p \\ &p^2+4p-480=0 \\ &(p-20)(p+24)=0 \\ &p=20 \text{ or } p=-24 \end{aligned}$$

Hence we conclude that the equilibrium price is \$20. Since the corresponding supply and demand are equal, we use the simpler demand equation to compute this quantity to get

$$D(20)=410-20=390$$

Hence, 390 units are supplied and demanded when the market is in equilibrium.

Exercises

១ ចូរគណនាតម្លៃអនុគមន៍ទៅតាមការបញ្ជាក់ដូចខាងក្រោម (Compute the indicated values of the given function)

a. $f(x)=3x^2+5x-2;\ f(1),\ f(0),\ f(-2)$

b. $g(x)=x+\dfrac{1}{x};\ g(-1),\ g(1),\ g(2)$

c. $h(t)=\sqrt{t^2+2t+4},\ h(2),\ h(0),\ h(-4)$

d. $f(t)=(2t-1)^{-3/2};\ f(1),\ f(5),\ f(13)$

e. $f(t)=\begin{cases} 3 & \text{if } t<-5 \\ t+1 & \text{if } -5\le t\le 5 \\ \sqrt{t} & \text{if } t>5 \end{cases};\ f(-6),\ f(-5),\ f(16)$

២ ចូរបញ្ជាក់អំពីដែនកំណត់របស់អនុគមន៍ដូចខាងក្រោម(Specify the domain of the given function)

a. $g(x)=\dfrac{x^2+1}{x+2}$ b. $y=\sqrt{x-5}$ c. $g(t)=\sqrt{t^2+9}$

d. $f(t)=(2t-4)^{3/2}$ e. $f(x)=\left(x^2-9\right)^{-1/2}$

៣ ឧបមាថាថ្លៃដើមសរុបគិតជាដុល្លាលើការផលិតទំនិញមួយប្រភេទ ចំនួនq ឯកតាត្រូវកំណត់ដោយអនុគមន៍ $C(q)=q^3-30q^2+400q+500$ ។

ក. គណនាថ្លៃដើមលើការផលិតចំនួន ២០ឯកតា ។

ខ. គណនាថ្លៃដើមលើការផលិតឯកតាទី២០ ។

(Suppose the total cost of manufacturing q units of a certain commodity is given by the function $C(q) = q^3 - 30q^2 + 400q + 500$. **a.** Compute the cost of manufacturing 20 units. **b.** Compute the cost of manufacturing the 20th unit.) (Answer: **a.** \$4,500 **b.** \$371)

៤ រកអនុគមន៍បណ្ដាក់ (Find the composite function) $g\left[h(x)\right]$

a. $g(u) = u^2 + 4,\ h(x) = x - 1$

b. $g(u) = 3u^2 + 2u - 6, \quad h(x) = x + 2$

c. $g(u) = (u-1)^3 + 2u^2, \ h(x) = x + 1$

d. $g(u) = \dfrac{1}{u^2}, \quad h(x) = x - 1$

e. $g(u) = u^2, h(x) = \dfrac{1}{x-1}$

៥ រកអនុគមន៍បណ្ដាក់តាមការបញ្ជាក់ដូចខាងក្រោម (find the indicated composite function)

a. $f(x+1)$ where $f(x) = x^2 + 5$

b. $f(x-2)$ where $f(x) = 2x^2 - 3x + 1$

c. $f(x-1)$ where $f(x) = (x+1)^5 - 3x^2$

d. $f\left(\dfrac{1}{x}\right)$ where $f(x) = 3x + \dfrac{2}{x}$

e. $f\left(x^2 + 3x - 1\right)$ where $f(x) = \sqrt{x}$

d. $f(x+1)$ where $f(x) = \dfrac{x-1}{x}$

៦ រកអនុគមន៍ $h(x)$ និង $g(u)$ ដែល $f(x) = g\left[h(x)\right]$

a. $f(x) = \left(x^5 - 3x^2 + 12\right)^3$

b. $f(x) = \sqrt{3x-5}$

c. $f(x) = (x-1)^2 + 2(x-1) + 3$

d. $f(x) = \sqrt{x+4} - \dfrac{1}{(x+4)^3}$

e. $f(x) = \sqrt{x+3} - \dfrac{1}{(x+4)^3}$

៧ នៅក្នុងរោងចក្រមួយ ថ្លៃដើមសរុបលើការផលិត q ឯកតាក្នុងកំលុងនៃសម្ញាក់ផលិតកម្មប្រចាំថ្ងៃ គឺ $C(q) = q^2 + q + 900$ ដុល្លា។ ក្នុងមួយថ្ងៃ បរិមាណផលិតផល $q(t) = 25t$ ឯកតាត្រូវបានផលិតក្នុងរយៈ ពេល t ដំបូងនៃខ្សែសង្វាក់ផលិតកម្ម។ **ក.** ចូរសរសេរ ថ្លៃដើមសរុបលើការផលិតជាអនុគមន៍នៃ t ។ **ខ.** តើនឹងត្រូវ ចំណាយប៉ុន្មានលើផលិតកម្ម នៅចុងម៉ោងទី៣? **គ.** តើនៅពេលណាដែលថ្លៃដើមសរុបលើការផលិតមានដល់ $\$11,000$? (At a certain factory, the total cost of manufacturing q units during the daily production run is $C(q) = q^2 + q + 900$ dollars. On a typical workday, $q(t) = 25t$ units are manufactured during the first t hours of a production run. **a.** Express the total manufacturing cost as a function of t. **b.**How much will have been spent on production by the end of the 3rd hour?) **c.** When will the total manufacturing cost reach \$11,000?

(Answer: a. $C\left[q(t)\right]=625t^2+25t+900$ b. \$6,600 c.After 4 hours)

៨ នៅក្នុងសេដ្ឋវិទ្យា ប្រាក់ចំណូលត្រូវបានកំណត់ថាជាទឹកប្រាក់ដែលបានពីការលក់ផលិតផល ហើយស្មើទៅនឹងផលគុណរវាងថ្លៃលក់ p ក្នុងមួយឯកតា និង ចំនួន x ឯកតានៃផលិតផលដែលបានលក់ ។ មានន័យថា $R=xp$ ។ ក្រឹត្យក្រមតម្រូវការ ចែងថា p និង x ទាក់ទងគ្នា ។ បើមួយកើនឡើង នោះមួយទៀតត្រូវថយចុះ ។ ឧបមាថា p និង x ទាក់ទងគ្នាតាមរយៈ **សមីការតម្រូវការ** $p=-\frac{1}{10}x+20,\ 0\le x\le 200$ ។ ចូរសរសេរកន្សោមប្រាក់ចំណូល R ជាអនុគមន៍នៃ x ។ (In economics, revenue R is defined as the amount of money derived from the sale of a product and is equal to the unit selling price p of the product times the number x of units actually sold. That is, $R=xp$. In economics, the Law of Demand states that p and x are related: As one increases, the other decrease. Suppose that p and x are related by the following demand equation: $p=-\frac{1}{10}x+20,\ 0\le x\le 200$. Express the revenue R as a function of the number x of units sold.) (Answer: $R(x)=-\frac{1}{10}x^2+20x$)

៩ ថ្លៃ p និងបរិមាណ x នៃផលិតផលមួយដែលបានលក់ គោរពទៅតាមសមីការតម្រូវការ

$$p=-\frac{1}{6}x+100, 0\le x\le 600$$ ។

ក. សរសេរកន្សោមប្រាក់ចំណូលជាអនុគមន៍ទៅនឹង x

ខ. បើគេលក់ផលិតផលអស់២០០ឯកតា តើប្រាក់ចំណូលស្មើប៉ុន្មាន?

គ. សង់ក្រាហ្វរបស់អនុគមន៍នេះ ។

ឃ. តើ x ស្មើប៉ុន្មានដែលនាំឱ្យប្រាក់ចំណូលអតិបរមា? តើតម្លៃអតិបរមានោះស្មើប៉ុន្មាន?

ង. រកតម្លៃលក់ដែលក្រុមហ៊ុនត្រូវកំណត់ ដើម្បីឱ្យប្រាក់ចំណូលអតិបរមា ។

(The price p and the quantity x sold of a certain product obey the demand equation

$$p=-\frac{1}{6}x+100, 0\le x\le 600$$

a. Express the revenue R as a function of x. (Remember, $R=xp$)
b. What is the revenue of the company if 200 units are sold?
c. Graph the revenue function.
d. What quantity x maximizes revenue? What is the maximum revenue?
e. What price should the company charge to maximize revenue?)

(Ans: **a.** $R(x)=-\frac{1}{6}x^2+100x$, **b.**\$13,333.33, **c.** 15,000 **d.** $x=300$, \$15,000 **e.** \$50)

១០ ថ្លៃ p និងបរិមាណ x នៃផលិតផលមួយដែលបានលក់ គោរពទៅតាមសមីការតម្រូវការ

$$x=-5p+100, 0\le p\le 20$$ ។

ក. សរសេរកន្សោមប្រាក់ចំណូលជាអនុគមន៍ទៅនឹង x

ខ. បើគេលក់ផលិតផលអស់១៥ឯកតា តើប្រាក់ចំណូលស្មើប៉ុន្មាន?

គ. សង់ក្រាហ្វរបស់អនុគមន៍នេះ ។

ឃ. តើ x ស្មើប៉ុន្មានដែលនាំឱ្យប្រាក់ចំណូលអតិបរមា? តើតម្លៃអតិបរមានោះស្មើប៉ុន្មាន?

ង. រកតម្លៃលក់ដែលក្រុមហ៊ុនត្រូវកំណត់ ដើម្បីឱ្យប្រាក់ចំណូលអតិបរមា ។

(The price p and the quantity x sold of a certain product obey the demand equation $x = -5p + 100, 0 \le x \le 20$

a. Express the revenue R as a function of x. (Remember, $R = xp$)

b. What is the revenue of the company if 15 units are sold?

c. Graph the revenue function.

d. What quantity x maximizes revenue? What is the maximum revenue?

e. What price should the company charge to maximize revenue?)

(Answer: **a.** $R(x) = -\frac{1}{5}x^2 + 20x$, **b.** \$255 **c.** 500 **d.** $x = 50$, \$500 **e.** \$10)

១១ ឧស្សាហករម្នាក់អាចផលិតម៉ាញ៉េក្នុងតម្លៃ២០ដុល្លាក្នុងមួយគ្រឿង ។ បើគេលក់ម៉ាញ៉េក្នុងតម្លៃ x ដុល្លាក្នុងមួយគ្រឿង នោះអតិថិជននឹងទិញ $120 - x$ គ្រឿងក្នុងមួយខែ ។ ចូរសរសេរកន្សោមប្រាក់ចំណេញប្រចាំខែរបស់ឧស្សាហករ ជាអនុគមន៍នឹងតម្លៃលក់ ។ សង់ក្រាហ្វរបស់អនុគមន៍ រួចតាមរយៈនេះ ចូរប៉ាន់ស្មានថ្លៃលក់ដែលប្រសើរបំផុត ។ (A manufacturer can produce cassette tape recorders at a cost of \$20 apiece. It is estimated that if the tape recorders are sold for x dollars a piece, consumers will buy $120 - x$ of them a month. Express the manufacturer's monthly profit as a function of price, graph this function, and use the graph to estimate the optimal selling price.)

(Answer: $P(x) = (x - 20)(120 - x)$, Optimal price: \$70 per recorder)

12 Write the equation for the line with the given properties

a. Through $(2, 0)$ with slope 1.

b. Through $(5, -2)$ with slope $\frac{-1}{2}$

c. Through $(2, 5)$ and parallel to the x axis

d. Though $(1, 0)$ and $(0, 1)$

e. Through $(2, 5)$ and $(1, 2)$

f. Through $(1, 5)$ and $(3, 5)$

១៣ ថ្លៃដើមសរុបរបស់ឧស្សាហករណ៏ម្នាក់ រួមមានចំណាយផ្សេងៗដែលថេរ គឺ ៥០០០ដុល្លាបូកនឹងថ្លៃដើមផលិតកម្មស្មើនឹង ៦០ ដុល្លាក្នុងមួយឯកតា ។ ចូរសរសេរកន្សោមថ្លៃដើមសរុប ជាអនុគមន៏នៃចំនួនឯកតាដែលបានផលិតរួចសង់ក្រាហ្វ ។ (A manufacturer's total cost consists of a fixed overhead of \$5,000 plus production costs of \$60 per unit. Express the total cost as a function of the number of units produced and draw the graph.). (Ans: $y = 60x + 5,000$)

១៤ លោកវេជ្ជបណ្ឌិតម្នាក់មានសៀវភៅផ្នែកវេជ្ជសាស្ត្រដែលមានតម្លៃ១៥០០ដុល្លា ដែលត្រូវបានកាត់រំលោះថេររហូតដល់សូន្យក្នុងរយៈពេលដប់ឆ្នាំ ។ មានន័យថា តម្លៃរបស់សៀវភៅទាំងនោះនឹងធ្លាក់ចុះក្នុងអត្រាថេររហូតដល់ វាស្មើសូន្យនៅចុងគ្រានៃរយៈពេលដប់ឆ្នាំ ។ ចូរសរសេរកន្សោមមួយតាងឱ្យតម្លៃរបស់សៀវភៅជាអនុគមន៏នៃ ពេល រួចសង់ក្រាហ្វ ។ (A doctor owns \$1,500 worth of medical books which, for tax purposes, are assumed to depreciate linearly to zero over a 10-year period. That is, the value of the books decreases at a constant rate so that it is equal to zero at the end of 10 years. Express the value of the books as a function of time and draw the graph.)(Answer: $y = -150x + 1,500$).

១៥ ចាប់តាំងពីដើមខែមក អាងរក្សាទឹកមួយបានបាត់បង់ទឹកក្នុងអត្រាថេរមួយ ។ នៅថ្ងៃទី១២ នៃខែ អាងរក្សាទឹកនោះផ្ទុកទឹកបាន ២០០ លាន ហ្គាឡុង ។ នៅថ្ងៃទី ២១ វានៅសល់តែ ១៦៤ លានហ្គាឡុង ។ ចូរសរសេរកន្សោមតាងឱ្យបរិមាណទឹកនៅក្នុងអាង ជាអនុគមន៍ទៅនឹងពេល រួចសង់ក្រាហ្វ ។ តើនៅថ្ងៃទី៨ ទឹកនៅក្នុងអាងមានចំណុះប៉ុន្មាន? (Since the beginning of the month, a local reservoir has been losing water at a constant rate. On the 12^{th} of the month the reservoir held 200 million gallons of water, and on the 21th it held only 164 million gallons. Express the amount of water in the reservoir as a function the time and draw the graph. How much water was in the reservoir as a function of time and draw the graph.)

(Answer: $y=-4x+248$ 216 million gallons).

១៦ ឧស្សាហករណ៏ម្នាក់អាចផលិតវិទ្យុដោយចំណាយលើថ្លៃដើម ២ដុល្លាក្នុងមួយគ្រឿង ។ វិទ្យុនោះត្រូវបានលក់ក្នុងតម្លៃ៥ដុល្លាក្នុងមួយគ្រឿង ដែលចំពោះតម្លៃនេះ អតិថិជននឹងទិញវិទ្យុទាំងនោះ៤០០០គ្រឿងក្នុងមួយខែ ។ ឧស្សាហករណ៏រូបនោះគ្រោងនឹងធ្វើការតម្លើងថ្លៃវិទ្យុទាំងនោះហើយបានរំពឹងទុកជាមុនថា សម្រាប់ការតម្លើងថ្លៃ ១ដុល្លា នោះចំនួនវិទ្យុដែលលក់ នឹងធ្លាក់ចុះចំនួន ៤០០គ្រឿងក្នុងមួយខែ ។ ចូរសរសេរកន្សោមតាងឱ្យចំនួនវិទ្យុ ដែលនឹងលក់ ដាច់ជាអនុគមន៍ទៅនឹងតម្លៃលក់ រួចរកមេគុណប្រាប់ទិស ។ **ណែនាំ:** ចំនួនវិទ្យុដែលនឹងលក់ ដាច់ y ជាអនុគមន៍លីនេអ៊ែរ ទៅ នឹងតម្លៃ x ដែលលក់ ហើយក្រាហ្វរបស់វាកាត់តាមចំណុច$(5,4000)$ ។(A manufacturer can produce radios at a cost of \$2 apiece. The radios have been selling for \$5 apiece, and at this price, consumers have been buying 4,000 radios a month. The manufacturer is planning to raise the price of the radios and estimates that for each \$1 increase in the price, 400 fewer radios will be sold each month. Express the number of radios sold as a function of the manufacturer's selling price. [*Hint:* Note that the number of radios sold *y* is a linear function of the selling price *x* and that its graph passes through the point (5, 4000). What is the slope?]

(Answer: $400(10-x), 0\le x\le 10$)

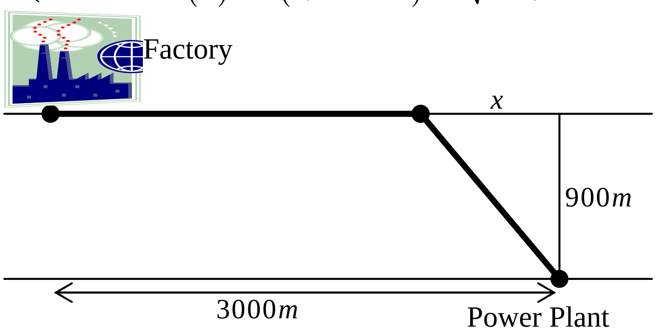
១៧ ក្នុងការផលិតទំនិញមួយឯកតា នោះគេចំណាយលើថ្លៃដើម $p=35x+15$ សេន នៅពេលដែល x ឯកតាត្រូវបានផលិត ។ បើគេលក់ទៅវិញក្នុងតម្លៃដដែលនេះ នូវទំនិញចំនួន x ឯកតា នោះចូរសរសេរអនុគន៏ប្រាក់ចំណូលជាអនុគមន៏នៃ x ។(Each unit of a certain commodity costs $p=35x+15$ cents when *x* units of the commodity are produced. If all *x* units are sold at this price, express the revenue derived from the sales as a function of *x*.)

(Ans: $P(x)=x(35x+15)$)

១៨ ផលិតករម្នាក់ លក់អំពូលភ្លើងក្នុងតម្លៃ៦ដុល្លាក្នុងមួយអំពូល ហើយចំពោះតម្លៃនេះ អតិថិជនបានទិញ៣០០០អំពូលក្នុងមួយខែ ។ ផលិតករនោះប្រាថ្នាចង់តម្លើងថ្លៃអំពូល ហើយបានរំពឹងទុកថា ចំពោះការឡើងថ្លៃ១ដុល្លានោះ វានឹងធ្វើឱ្យចំនួនអំពូលដែលលក់ដាច់ ធ្លាក់ចុះចំនួន១០០០អំពូលក្នុងមួយខែ ។ ផលិតករនោះអាចផលិតអំពូលដោយប្រើថ្លៃដើម៤ដុល្លាក្នុងមួយអំពូល ។ ចូរសរសេរកន្សោមប្រាក់ចំណេញប្រចាំខែរបស់ផលិតករពីការលក់អំពូល ជាអនុគមន៍ទៅនឹងថ្លៃលក់ សង់ក្រាហ្វ រួចប៉ាន់ស្មានថ្លៃលក់ដែលប្រសើរបំផុត ។(A manufacturer has been selling lamps at the price of \$6 per lamp, and at this price, consumers have been buying 3,000 lamps a month. The manufacturer wishes to raise the price and estimates that for each $1 increase in the price, 1,000 fewer lamps will be sold each month. The manufacturer can produce the lamps at a cost of $4 per lamp. Express the manufacturer’s monthly profit as a function of the price that the lamps are sold, draw the graph, and estimate the optimal selling price.)(Answer: $6.50)

១៩ គេត្រូវភ្ជាប់ខ្សែកាបពីរោងចក្រអគ្គិសនីពីមាត់ទន្លេម្ខាងទៅរោងចក្រមួយដែលស្ថិតនៅត្រើយម្ខាងទៀតដែលស្ថិតនៅក្រោមខ្សែទឹកពីរោងចក្រអគ្គិសនីចម្ងាយ៣000ម៉ែត្រហើយប្រវែងទទឹងរបស់ទន្លេនោះគឺ៩00ម៉ែត្រ ។ ថ្លៃខ្សែកាបដែលភ្ជាប់ឆ្លងកាត់ទន្លេស្មើនឹង ៥ដុល្លា ក្នុងមួយម៉ែត្រ រីឯថ្លៃខ្សែកាបដែលភ្ជាប់លើដីស្មើ នឹង ៤ដុល្លាក្នុងមួយម៉ែត្រ។ សូមមើលរូបបន្ថែម។ ចូរសរសេរតម្លៃចំណាយលើការតម្លើងខ្សែកាបជាអនុគមន៍ទៅនឹង x ។ (A cable is to be run from a power plant on one side of a river 900 meters wide to a factory on the other side, 3,000 meters downstream. The cable will be run in a straight line from the power plant to some point P on the opposite bank and then along the bank to the factory. The cost of running the cable across the water is $5 per meter, while the cost over land is $4 per meters. Let x be the distance from P to the point directly across the river from the power plant and express the cost of installing the cable as a function of x.)

(Answer: $C(x)=4(3,000-x)+5\sqrt{810,000+x^2}$)



២០ ផលិតករគ្រឿងសង្ហារឹមម្នាក់លក់តុមួយក្នុងតម្លៃ ៧0ដុល្លា។ ថ្លៃដើមសរុបរួមមានចំណាយផ្សេងៗថេរ ៨000 ដុល្លា និងថ្លៃដើមផលិតកម្ម ៣0ដុល្លាចំពោះតុមួយ ។

ក. តើគាត់ត្រូវលក់តុចំនួនប៉ុន្មានដើម្បីឱ្យរួចដើម?

ខ.តើគាត់ត្រូវលក់តុចំនួនប៉ុន្មានដើម្បីឱ្យបានប្រាក់ចំណេញ ៦000ដុល្លា?

គ. បើលក់ដាច់ ១៥0 តុ តើគាត់ខាតប៉ុន្មាន និងចំណេញប៉ុន្មាន?

ឃ. នៅលើប្រព័ន្ធអ័ក្សតែមួយ ចូរសង់ក្រាហ្វនៃអនុគមន៍ប្រាក់ចំណូល និងអនុគមន៍ថ្លៃដើម ។

(A furniture manufacturer can sell dining-room tables for $70 apiece. The manufacture‘s total cost consists of a fixed overhead of $8,000 plus production costs of $30 per table.

a. How many tables must be the manufacturer sell to break even?
b. How many tables must the manufacturer sell to make a profit of $6,000?
c. What will the manufacturer’s profit or loss if 150 tables are sold?
d. On the same axes, graph the manufacture’s total revenue and total cost functions.

២១ ចាប់តាំងពីដើមឆ្នាំមក តម្លៃប្រេងសាំងកើនឡើងក្នុងអត្រាថេរស្មើនឹង 2 សេនក្នុងមួយហ្គាឡុងក្នុងមួយខែ ។ គិតត្រឹមថ្ងៃទី១មិថុនា តម្លៃនេះឡើងដល់ 1.03 ដុល្លាក្នុងមួយហ្គាឡុង ។

ក. ចូរសរសេរអនុគមន៍តម្លៃប្រេងសាំងជាអនុគមន៍ទៅនឹងពេល រួចសង់ក្រាហ្វ ។

ខ. តើកាលពីនៅដើមឆ្នាំ តម្លៃប្រេងសាំងស្មើនឹងប៉ុន្មាន?

គ. តើនៅថ្ងៃទី១ខែតុលា តម្លៃប្រេងសាំងស្មើនឹងប៉ុន្មាន?

(Since the beginning of the year, the price gasoline has been increasing a constant rate of 2 cents per gallon per month. By June first, the price had reached $1.03 per gallon.

a. Express the price of the gasoline as a function of time and draw the graph.
b. What was the price at the beginning of the year?
c. What will the price be on October first?)

(Answer: **a.** $P(x) = 2x + 93$, **b.** 93cents/gallon, **c.** $1.11 per gallon)

Chapter 2

Differentiation: Basic Concepts

1 The Derivative

Definition

For the function $y = f(x)$, the derivative of f at x is defined to be:

$$f'(x) = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

provided that the limit exists.

To find a derivative by using the definition

1. Form the ratio $\frac{f(x+\Delta x) - f(x)}{\Delta x}$, called the **difference quotient**.
2. Simplify the difference quotient algebraically.
3. Calculate $f'(x) = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$

Example 1

Use the definition of derivative to find $f'(x)$ for $f(x) = x^2$.

Solution

Step 1: Form the deference quotient.

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

Step 2: Simplify the difference quotient

$$\frac{(x+\Delta x)^2 - x^2}{\Delta x} = \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = 2x + \Delta x$$

Step 3: Find the limit.

$$\lim_{\Delta x \to 0}(2x + \Delta x) = 2x + 0 = 2x$$

Therefore, $f'(x) = 2x$.

Example 2

Suppose a manufacturer's profit from the sale of radios is given by the function $P(x) = 400(15 - x)(x - 2)$, where x is the price at which the radios are sold. Find the selling price that maximizes profit.

Solution

Your goal is to find the value of x that maximizes the profit $P(x)$. This is the value of x for which the slope of the tangent line is zero. Since the slope of the tangent line is given by the derivative, begin by computing $P'(x)$.For simplicity; apply the definition of the derivative to the unfactored form of the profit function.

$$P(x) = -400x^2 + 6,800x - 12,000$$

You find that

$$P'(x) = \lim_{\Delta x \to 0}\left[\frac{P(x+\Delta x) - P(x)}{\Delta x}\right]$$

then

$$P'(x) = \lim_{\Delta x \to 0}\frac{-400(x+\Delta x)^2 + 6,800(x+\Delta x) - 12,000 - \left(-400x^2 + 68,00x - 12,000\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\frac{-400(\Delta x)^2 - 800x\Delta x + 6,800\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\left(-400\Delta x - 800x + 6,800\right) = -800x + 6,800$$

To find the value of x for which the slope of the tangent is zero, set the derivative equal to zero and solve the resulting equation for x as follows:

$$P'(x) = 0$$

then

$$-800x + 6800 = 0$$

$$800x = 6800$$

$$x = \frac{6800}{800} = 8.5$$

It follows that $x = 8.5$ are the x coordinates of the peak of the graph and that the optimal selling price is \$ 8.50 per radio.

2 Techniques of Differentiation

2.1 The Power Rule

For any number *n*,

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

That is, to find the derivative of x^n, reduce the power of x by 1 and multiply by the original power.

Example 1

Differentiate (find the derivative of) each of the following functions:

a. $y = x^{27}$, b. $y = \frac{1}{x^{27}}$, c. $y = \sqrt{x}$, d. $y = \frac{1}{\sqrt{x}}$

Solution

In each case, use exponential to write the function as a power function and then apply the general rule.

a. $\frac{d}{dx}\left(x^{27}\right) = 27x^{27-1} = 27x^{26}$

b. $\frac{d}{dx}\left(\frac{1}{x^{27}}\right) = \frac{d}{dx}\left(x^{-27}\right) = -27x^{-27-1} = -27x^{-28}$

c. $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

d. $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}\left(x^{-\frac{1}{2}}\right) = -\frac{1}{2}\left(x^{-\frac{1}{2}-1}\right) = -\frac{1}{2}\left(x^{-\frac{3}{2}}\right) = -\frac{1}{2\sqrt{x^3}}$

2.2 The Derivative of a constant

For any constant *C*,

$$\frac{d}{dx}(C)=0$$

That is, the derivative of a constant is zero.

2.3 The Constant Multiple Rule

For any constant *C*,

$$\frac{d}{dx}(Cf)=C\frac{df}{dx}=Cf'(x)$$

That is, the derivative of a constant time a function is equal to the constant times the derivative of the function.

Example 2

Differentiate the function $y=3x^5$

Solution:

You already know that $\frac{d}{dx}\left(x^5\right)=5x^4$. Combining this with the constant multiple rule, you get $\frac{d}{dx}\left(3x^5\right)=3\frac{d}{dx}\left(x^5\right)=3\left(5x^4\right)=15x^4$.

2.4 The Sum Rule

$$\frac{d}{dx}(f\pm g)=\frac{df}{dx}\pm\frac{dg}{dx}=f'(x)\pm g'(x)$$

That is, the derivative of a sum is the sum of the individual derivatives.

Example 3

Differentiate the function $y=x^2+3x^5$

Solution

$$\frac{d}{dx}\left(x^2+3x^5\right)=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}\left(3x^5\right)=2x+15x^4$$

2.5 The Product Rule

$$\frac{d}{dx}(fg)=g\frac{df}{dx}+f\frac{dg}{dx}=f'(x)g(x)+g'(x)f(x)$$

That is, the derivative of a product is the first factor times the derivative of the second plus the second factor times the derivative of the first.

Example 4

Differentiate the function $y=x^2(3x+1)$

Solution

According to the product rule

$$\begin{aligned}\frac{d}{dx}\left[x^2(3x+1)\right]&=(3x+1)\frac{d}{dx}\left(x^2\right)+x^2\frac{d}{dx}(3x+1)\\&=(3x+1)2x+x^2\times 3\\&=9x^2+2x\end{aligned}$$

2.6 The Derivative of a Quotient

$$\frac{d}{dx}\left(\frac{f}{g}\right)=\frac{g\dfrac{df}{dx}-f\dfrac{dg}{dx}}{g^2}$$

Example 5
Differentiate the rational function

$$y=\frac{x^2+2x-21}{x-3}$$

Solution
According to the quotient rule,

$$\frac{dy}{dx}=\frac{(x-3)\dfrac{d}{dx}\left(x^2+2x-21\right)-\left(x^2+2x-21\right)\dfrac{d}{dx}(x-3)}{(x-3)^2}$$

$$=\frac{(x-3)(2x+2)-\left(x^2+2x-21\right)}{(x-3)^2}$$

$$=\frac{2x^2-4x-6-x^2-2x+21}{(x-3)^2}=\frac{x^2-6x+15}{(x-3)^2}$$

3 The Derivative as a Rate of change

3.1 Average and Instantaneous Rate of Change

Suppose that *y* is a function of *x*, say $y=f(x)$. Corresponding to a change from *x* to $x+\Delta x$, the variable y changes by an amount $\Delta y=f(x+\Delta x)-f(x)$. The resulting **average rate of change of *y* with respect to *x*** is the difference quotient

$$\text{Average rate of change}=\frac{\text{change in } y}{\text{change in } x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

As the interval over which you are averaging becomes shorter (that is, as Δx approaches zero), the average rate of change approaches what you would intuitively call the **instantaneous rate of change of *y* with respect to *x*** and the difference quotient approaches the derivative $f'(x)$ or $\dfrac{dy}{dx}$. That is,

$$\text{Instantaneuous rate of change}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=f'(x)=\frac{dy}{dx}$$

Instantaneous Rate of Change
If $y=f(x)$, the instantaneous rate of change of *y* with respect to x is given by the derivative of *f*. That is,

$$\text{Rate of change}=f'(x)=\frac{dy}{dx}$$

Example 1
It is estimated that *x* months from now, the population of a certain community will be

$$P(x)=x^2+20x+8{,}000$$

a. At what rate will the population be changing with respect to time 15 months from now?
b. By how much will the population actually change during the 16th month?

Solution

a. The rate of change of the population with respect to time is the derivative of the population function. That is,

$$\text{Rate of change} = P'(x) = 2x + 20$$

The rate of change of the population 15 months from now will be

$$P'(15) = 2\times 15 + 20 = 50 \text{ people per month}$$

b. The actual change in the population during the 16th month is the difference between the population at the end of 16 month and the population at the end of 15 months. That is,

$$\text{Change in population} = P(16) - P(15) = 8,576 - 8,525 = 51 \text{ People}$$

3.2 Percentage Rate of Change

$$\text{Percentage rate of change} = 100\frac{\text{rate of change of quantity}}{\text{size of quantity}}$$

If $y = f(x)$, the percentage rate of change of y with respect to x is given by the formula

$$\text{Percentage rate of change} = 100\frac{f'(x)}{f(x)} = 100\frac{dy/dx}{y}$$

Example 2

The gross national product (GNP) of a certain country was $N(t) = t^2 + 5t + 106$ billion dollars years after 1980.

a. At what rate was the GNP changing with respect to time in 1988?
b. At what percentage rate was the GNP changing with respect to time in 1988?

Solution

a. The rate of change of the GNP is the derivative $N'(t) = 2t + 5$. The rate of change in 1988 was $N'(8) = 2\times 8 + 5 = 21$ billion dollars per year.

b. The percentage rate of change of the GNP in 1988 was

$$100\frac{N'(8)}{N(8)} = 100\frac{21}{210} = 10 \text{ percent per year}$$

4 Approximation by Differentials; Marginal Analysis

If y is a function of x, then we have basic formula

$$\begin{pmatrix}\text{Change}\\ \text{in } y\end{pmatrix} \approx \begin{pmatrix}\text{rate of change of } y\\ \text{with respect to } x\end{pmatrix} \times (\text{change in } x)$$

Approximation Formula

If $y = f(x)$ and Δx is a small change in x, then the corresponding change in y is

$$\Delta y \approx \frac{dy}{dx}\Delta x$$

or, in functional notation, the corresponding change in f is

$$\Delta f = f(x + \Delta x) - f(x) \approx f'(x)\Delta x$$

That is, the change in the function is approximately that derivative of the function times the change in its variable.

Example 1

Suppose the total cost in dollars of manufacturing q units of a certain commodity is $C(q) = 3q^2 + 5q + 10$ if the current level of production is 40 units, estimate how the total cost will change if 40.5 units are produced.

Solution

In this problem, the current value of the variable is $q = 40$ and the change in the variable is $\Delta q = 0.5$, by the approximation formula, the corresponding change in cost is

$$\Delta C = C(40.5) - C(40) \simeq C'(40)\Delta q = C'(40) \times 0.5$$

since

$$C'(q) = 6q + 5 \text{ and } C'(40) = 6 \times 40 + 5 = 245$$

it follows that

$$\Delta C \simeq C'(40) \times 0.5 = 245 \times 0.5 = \$122.50$$

Example 2

The daily output at a certain factory is $Q(L) = 900L^{1/3}$ units where L denotes the size of the labor force measured in worker-hours. Currently, 1,000 worker-hours of labor are used each day. Use calculus to estimate the number of additional worker-hours of labor that will be needed to increase daily output by 15 units.

Solution

Solve for ΔL using approximating formula

$$\Delta Q \simeq Q'(L) \times \Delta L$$

with

$$\Delta Q = 15,\ L = 1{,}000 \text{ and } Q'(L) = 300L^{-2/3}$$

to get

$$15 \approx 300(1{,}000)^{-2/3}\,\Delta L$$

or

$$\Delta L \approx \frac{15}{300}(1{,}000)^{2/3} = \frac{15}{300} \times 10^2 = 5 \text{ worker-hours}$$

4.1 Approximation of Percentage change

The **percentage change** of a quantity expresses the change in that quantity as a percentage of its size prior to the change. In particular,

$$\text{Percentage change} = 100\,\frac{\text{change in quantity}}{\text{size of quantity}}$$

This formula can be combined with the approximation formula and written in functional notation as follows.

Approximation Formula for Percentage Change

If Δx is a (small) change in x, the corresponding percentage change in the function $f(x)$ is

$$\text{Percentage change in } f = 100 \times \frac{\Delta f}{f(x)} \simeq 100 \times \frac{f'(x) \times \Delta x}{f(x)}$$

Example 3

The GNP of a certain country was $N(t) = t^2 + 5t + 200$ billion dollars t years after 1990. Use calculus to estimate the percentage change in the GNP during the first quarter of 1998.

Solution

Use the formula

$$\text{Percentage change in } N \simeq 100\frac{N'(t)\Delta t}{N(t)}$$

With $t = 8$, $\Delta t = 0.25$ and $N'(t) = 2t + 5$ to get

$$\begin{aligned}\text{Percentage change in } N &\simeq 100\frac{N'(8)\times 0.25}{N(8)}\\ &\simeq 100\frac{(2\times 8+5)0.25}{8^2+5\times 8+200}\\ &\simeq 1.73 \text{ percent}\end{aligned}$$

Example 4

At a certain factory, the daily outputis $Q(K) = 4,000K^{1/2}$ units, where K denotes the firm's capital investment. Use calculus to estimate the percentage increase in output that will result from a 1 percent increase in capital investment.

Solution

The derivative is $Q'(K) = 2,000K^{-1/2}$. The fact that K increases by 1 percent means that $\Delta K = 0.01K$. Hence,

$$\begin{aligned}\text{Percentage change in } Q &\simeq 100\frac{Q'(K)\Delta K}{Q(K)}\\ &= 100\frac{2,000k^{-1/2}(0.01K)}{4,000K^{1/2}}\\ &= 0.5 \text{ percent}\end{aligned}$$

4.2 Marginal Analysis in Economics

In economics, the use of the derivative to approximate the change in a function produced by a 1-unit change in its variable is called **marginal analysis**. For example, if $C(x)$ is the total production cost incurred by a manufacturer when x units are produced and $R(x)$ is the total revenue derived from the sale of x units, then $C'(x)$ is called the **marginal cost** and $R'(x)$ is called the **marginal revenue**. If production (or sales) is increased by 1 unit, then $\Delta x = 1$ and the approximation formula:

$$\Delta C = C(x+\Delta x) - C(x) \approx C'(x)\Delta x$$

becomes

$$\Delta C = C(x+1) - C(x) \approx C'(x)$$

while

$$\Delta R = R(x+\Delta x) - R(x) \approx R'(x)\Delta x$$

becomes

$$\Delta R = R(x+1) - R(x) \approx R'(x)$$

That is, the marginal cost $C'(x)$ is an approximation to the cost $C(x+1)-C(x)$ of producing the $(x+1)$st unit, and similarly, the marginal revenue $R'(x)$ is an approximation to the revenue derived from the sale of the $(x+1)$st unit. To summarize:

Approximation by Marginal cost and Marginal Revenue

If $C(x)$ is the total cost of producing *x* units and $R(x)$ is the total revenue derived from the sale of *x* units, then

Marginal cost $= C'(x) \approx$ the cost of producing the $(x+1)$st unit

Marginal revenue $= R'(x) \approx$ the revenue derived from the sale of the $(x+1)$st unit

Example 5

A manufacture estimates that when *x* units of a particular commodity are produced, the total cost will be $C(x) = \frac{1}{8}x^2 + 3x + 98$ dollars, and that $P(x) = \frac{1}{3}(75-x)$ dollars per unit is the price at which all *x* units will be sold.

a. Find the marginal cost and the marginal revenue.
b. Use marginal cost to estimate the cost of producing the 9^{th} unit.
c. What is the actual cost of producing the 9^{th} unit?
d. Use the marginal revenue to estimate the revenue derived from the sale of the 9^{th} unit.
e. What is the actual revenue derived from the sale of the 9^{th} unit?

Solution

a. The marginal cost is $C'(x) = \frac{1}{4}x + 3$. Since *x* units of the commodity are sold at a price of $P(x) = \frac{1}{3}(75-x)$ dollars per unit, the total revenue is

$$R(x) = (\text{number of units sold}) \times (\text{price per unit})$$

$$= xP(x) = x\left[\frac{1}{3}(75-x)\right] = 25x - \frac{1}{3}x^2$$

The marginal revenue is $R'(x) = 25 - \frac{2}{3}x$

b. The cost of producing the 9^{th} unit is the change in cost as *x* increase from 8 to 9 and can be estimated by the marginal cost

$$C'(x) = \frac{1}{4} \times 8 + 3 = \$5$$

c. The actual cost of producing the 9^{th} unit is

$$\Delta C = C(9) - C(8) = \$5.13$$

which is reasonable well approximated by the marginal cost $C'(8) = \$5$

d. The revenue obtained from the sale of the 9^{th} unit is approximated by the marginal revenue:

$$R'(8) = 25 - \frac{2}{3}\times 8 = \$19.67$$

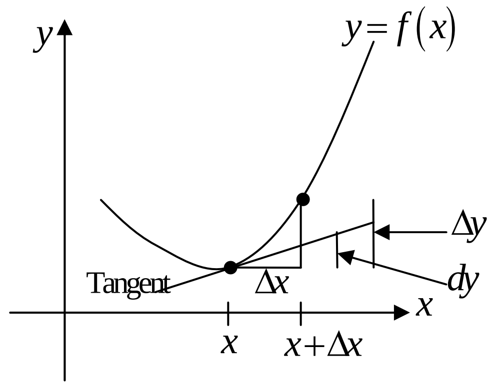
e. The actual revenue obtained from the sale of the 9th unit is

$$\Delta R = R(9) - R(8) = \$19.33$$

4.3 Differentials

The expression $f'(x)\Delta x$ on the right hand side of the approximation formula $\Delta f \simeq f'(x)\Delta x$ is sometimes called the **differential** of *f* and is denoted by *df*.

Similarly, the expression $\frac{dy}{dx}\Delta x$ on the right-hand side of the other form of the approximation formula $\Delta y \simeq \frac{dy}{dx}\Delta x$ is known as the differential of *y* and is denoted by *dy*.



Thus, is Δx is small,

$$\Delta y \simeq dy \text{ where } dy = \frac{dy}{dx}\Delta x$$

5 The Chain Rule

Suppose the total manufacturing cost at a certain factory is a function of the number of units produced, which in turn is a function of the number of hours during which the factory has been operating. Let *C*, *q* and *t* denote the cost (in dollars), the number of units, and the number of hours, respectively. Then,

$$\frac{dC}{dq} = \text{ rate of change with respect to output (dollars per unit)}$$

and

$$\frac{dq}{dt} = \text{rate of change of output with respect to time (units per hour)}$$

The product of these two rates is the rate of change of cost with respect to time. That is,

$$\frac{dC}{dq}\frac{dq}{dt} = \begin{pmatrix}\text{rate of change of cost} \\ \text{with respect to time}\end{pmatrix} \text{ (dollars per hours)}$$

Since the rate of change of cost with respect to time is also given by the derivative $\frac{dC}{dt}$, it follows that

$$\frac{dC}{dt} = \frac{dC}{dq}\frac{dq}{dt}$$

This formula is a special case of an important rule called the **chain rule**.

The Chain Rule

Suppose *y* is a differentialbe function of *u* and *u* is a differentiable function of *x*. Then *y* can be regarded as a function *x* and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example 1

Suppose that $y = u + \sqrt{u}$ and $u = x^3 + 17$. Use the Chain Rule to find $\frac{dy}{dx}$, then evaluate $\frac{dy}{dx}$ at x=2 (Such an evaluation is denoted by $\left.\frac{dy}{dx}\right|_{x=2}$)

Solution

$y = u + \sqrt{u} = u + u^{1/2}$ and $u = x^3 + 17$

$$\frac{dy}{du} = 1 + \frac{1}{2}u^{-\frac{1}{2}} = 1 + \frac{1}{2\sqrt{u}} \quad \text{and} \quad \frac{du}{dx} = 3x^2$$

So, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(1 + \frac{1}{2\sqrt{u}}\right)3x^2$$

If $x = 2$, then $u = 2^3 + 17 = 25$ and $\frac{du}{dx} = 3 \times 2^2 = 12$. Hence

$$\left.\frac{dy}{dx}\right|_{x=2} = \left(1 + \frac{1}{2\sqrt{25}}\right)\times 12 = \left(1 + \frac{1}{10}\right)\times 12 = \frac{66}{5}$$

Example 2

Find $\frac{dy}{dx}$ when $x = 1$ if $y = \frac{u}{u+1}$ and $u = 3x^2 - 1$ (Answer: 2/3)

Example 3

Differentiate the function

a. $f(x) = \sqrt{x^2 + 3x + 2}$ b. $f(x) = \left(2x^4 - x\right)^3$ c. $f(x) = \frac{1}{\left(2x+3\right)^5}$

Example 4

An environmental study of a certain suburban community suggests that the average daily level of carbon monoxide in the air will be $C(p) = \sqrt{0.5p^2 + 17}$ parts per million when the population is p thousand. It is estimated that t years from now, the population of the community will be $p(t) = 3.1 + 0.1t^2$ thousand.

At what rate will the carbon monoxide level be changing with respect to time 3 years from now?

Solution

The goal is to find $\frac{dC}{dt}$ when $t = 3$. Since

$$\frac{dC}{dp} = \frac{1}{2}\left(0.5p^2 + 17\right)^{-\frac{1}{2}}\left[0.5\left(2.p\right)\right] = \frac{1}{2}p\left(0.5p^2 + 17\right)^{-\frac{1}{2}}$$

and

$$\frac{dp}{dt} = 0.2t$$

it follows from the chain rule that

$$\frac{dc}{dt} = \frac{dc}{dp}\times\frac{dp}{dt} = \frac{1}{2}p\left(0.5p^2 + 17\right)^{-\frac{1}{2}}\left(0.2t\right) = \frac{0.1pt}{\sqrt{0.5p^2 + 17}}$$

when $t=3$, $p=p(3)=3.1+0.1\times3^2=4$ and so

$$\frac{dc}{dt}=\frac{0.1\times4\times3}{\sqrt{0.5\times4^2+17}}=\frac{1.2}{\sqrt{25}}=0.24 \text{ parts per million per year}$$

6 Higher-Order Derivatives

6.1 The Second Derivative

The second derivative of a function is the derivative of its derivative. If $y=f\left(x\right)$, the second derivative is denoted by:

$$\frac{d^2y}{dx^2} \text{ or } f''\left(x\right)$$

The second derivative gives the rate of change of the rate change of the original function.

Example 1

Find both the first and second derivatives of the functions:

a. $f\left(x\right)=x^3-12x+1$ b. $f\left(x\right)=5x^4-3x^2-3x+7$ c. $f\left(x\right)=\dfrac{3x-2}{\left(x-1\right)^2}$

Example 2

An efficiency study of the morning shift at a certain factory indicates that an average worker who arrives on the job at 8:00AM. Will have produced $Q\left(t\right)=-t^3+6t^2+24t$ units t hours later.

- **a.** Compute the worker's rate of production at 11:00A.M
- **b.** At what rate is the worker's rate of production changing with respect to time at 11:00A.M?
- **c.** Use calculus to estimate the change in the worker's rate of production between 11:00 and 11:10A.M.
- **d.** Compute the actual change in the worker's rate of production between 11:00 and 11:10A.M.

Solution

- **a.** The worker's rate of production is the first derivative

  $$Q'\left(t\right)=-3t^2+12t+24$$

  At 11:00 A.M., $t=3$ and the rate of production is

  $$Q'\left(3\right)=-3\times3^2+12\times3+24=33 \text{ units per hour}$$

- **b.** The rate of change of the rate of production is the second derivative

  $$Q''\left(t\right)=-6t+12$$

  At 11:00 A.M., the rate is

  $$Q''\left(3\right)=-6\times3+12=-6 \text{ unit per hour per hour}$$

- **c.** Note that 10 minutes is 1/6 hours, and hence $\Delta t=1/6$ hour.

  Change in rate of production is $\Delta Q'\simeq Q''\left(t\right)\Delta t$

  $$=-6\times\frac{1}{6}=-1 \text{ unit per hour}$$

d. The actual change in the worker's rate of production between 11:00 and 11:10 A.M. is the difference between the values of the rate $Q'(t)$ when $t = 3$ and when $t = 3\frac{1}{6} = \dfrac{19}{6}$. That is,

$$\begin{pmatrix}\text{Actual change in}\\ \text{rate of production}\end{pmatrix} = Q'\left(\frac{19}{6}\right) - Q'(3) = -1.08 \text{ units per hour}$$

6.2 The nth Derivative

For any positive integer *n*, the nth derivative of a function is obtained from the function by differentiating successively *n* times. If the original function is $y = f(x)$, the n^{th} derivative is denoted by

$$\frac{d^n y}{dx^n} \quad \text{or} \quad f^{(n)}(x)$$

Example 3

Find the 5th derivative of each of the following functions:

a. $f(x) = 5x^6 + 2x^4 + x^2 - 3$ b. $y = \dfrac{1}{x}$.

7 Concavity and the Second Derivative Test

Concavity

Suppose that *f* is differentiable on the interval (*a*,*b*).

a. If f' is increasing on (*a*,*b*),then the graph of f is **concave upward** on (a,b).

b. If f' is decreasing on (*a*,*b*), then the graph of f is **concave downward** on (*a*,*b*).

To Determine Concavity

Suppose that f is a function and f' and f'' both exist on the interval (*a*,*b*).

a. If $f''(x) > 0$ for all *x* in (a,b), then f' is increasing and f is concave upward on (*a*,*b*).

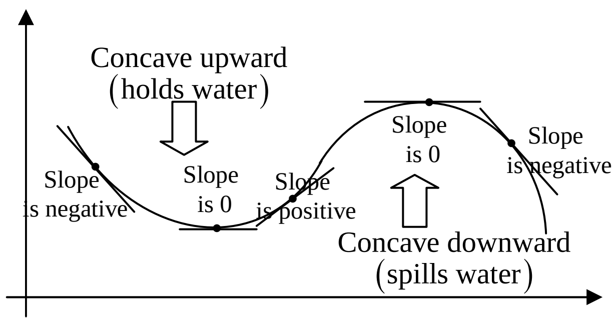
b. If $f''(x) < 0$ for all *x* in (*a*,*b*), then f' is decreasing and f is concave downward on (*a*,*b*).

Critical Points

A critical point of a function is a point on its graph where either:

+ The derivative is zero, or
+ The derivative is undefined

The relative maxima and minima of the function can occur only at critical points.



To Determine Points of Inflection

A point on the graph of a function at which the concavity of the function changes is called an **inflection point.**

Suppose that f is a continuous function.

a. Find f''

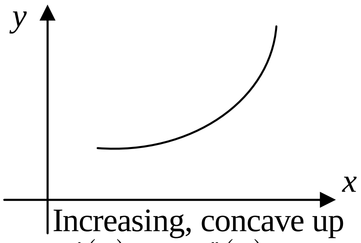
b. Find the **hypercritical values** of x. That is, find the values $x=c$ where

+ $f''(c)=0$, or
+ $f''(c)$ is undefined

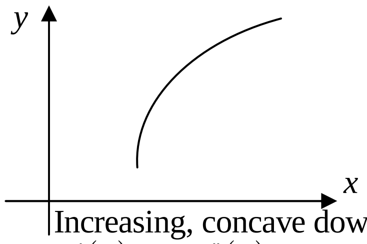
c. Using the **hypercritical values** as endpoints of intervals, determine the interval where

+ $f''(x)>0$ and f is concave upward, and
+ $f''(x)<0$ and f is concave downward.

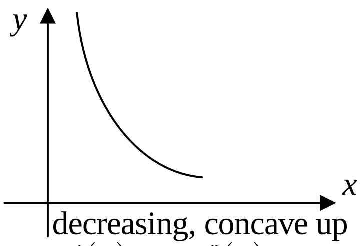
d. Point of inflection occurs at those hypercritical values where f changes concavity.



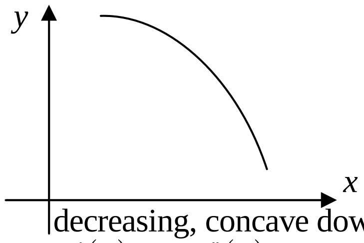
Increasing, concave up
$f'(x)>0,\ f''(x)>0$



Increasing, concave down
$f'(x)>0,\ f''(x)<0$



decreasing, concave up
$f'(x)<0,\ f''(x)>0$



decreasing, concave down
$f'(x)<0,\ f''(x)<0$

Example 1

For the function $f(x)=x^3-\frac{3}{2}x^2+5$ (a) determine the intervals on which f is concave upward and the intervals on which it is concave downward, and (b) locate any points of inflection.

Solution

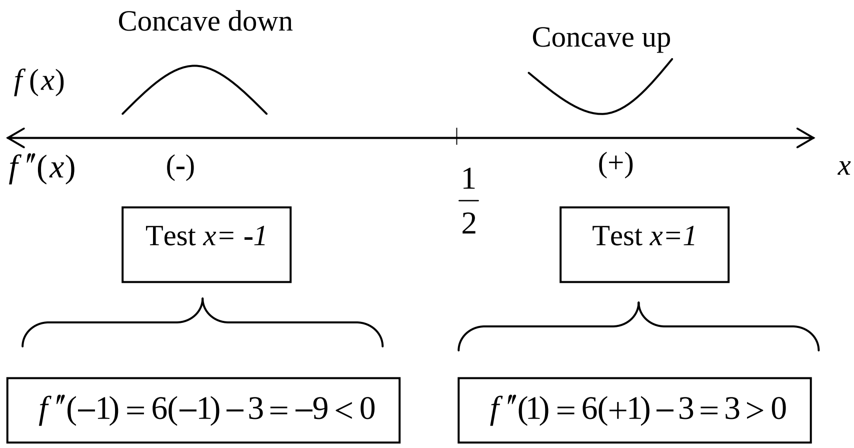
a. We find the hypercritical values and then determine the concavity on the related intervals.

$$f(x)=x^3-\frac{3}{2}x^2+5$$
$$f'(x)=3x^2-3x$$
$$f''(x)=6x-3$$

Now set $f''(x)=0$ (There are no values where $f''(x)=0$ is undefined.)

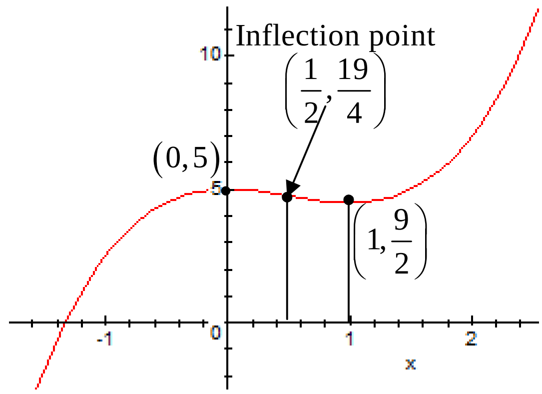
$$6x-3=0,\ x=\frac{1}{2}$$

There are two intervals to be considered: $\left(-\infty,\frac{1}{2}\right)$ and $\left(\frac{1}{2},+\infty\right)$



f is concave upward on $\left(\frac{1}{2}, +\infty\right)$ and downward on $\left(-\infty, \frac{1}{2}\right)$

b. $f\left(\frac{1}{2}\right) = \frac{19}{4}$ changes concavity at $x = \frac{1}{2}$, therefore the point $\left(\frac{1}{2}, \frac{19}{4}\right)$ is a point of inflection.



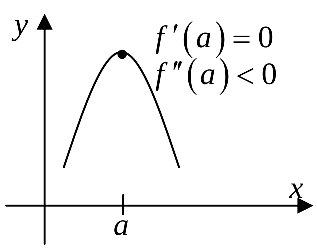
Second-Derivative Test

Suppose that $f'(a) = 0$.

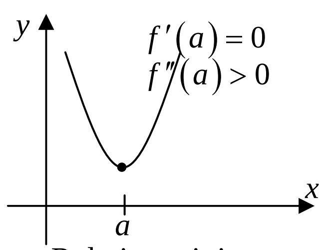
If $f''(a) > 0$, then f has a relative minimum at $x = a$.

If $f''(a) < 0$, then f has a relative maximum at $x = a$.

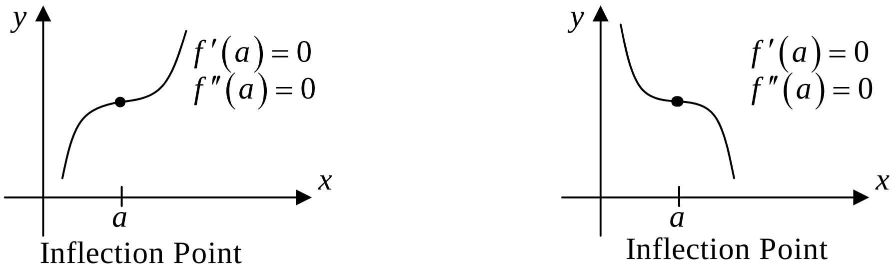
However, if $f''(a) = 0$, the test is inconclusive and f may have a relative maximum, a relative minimum, or no relative extremum all at $x = a$.



Relative maximum



Relative minimum



The behaviou of a graph when the first derivative is zero.

Example 2

Locate the local extrema for the function $f(x)=\frac{8}{3}x^3-x^4$.

Solution

We Find $f'(x)$ and all values $x=a$ where $f'(a)=0$.

$$f(x)=\frac{8}{3}x^3-x^4$$
$$f'(x)=8x^2-4x^3$$

$$8x^2-4x^3=0$$
$$4x^2(2-x)=0$$
$$x=0 \quad \text{or} \quad x=2$$

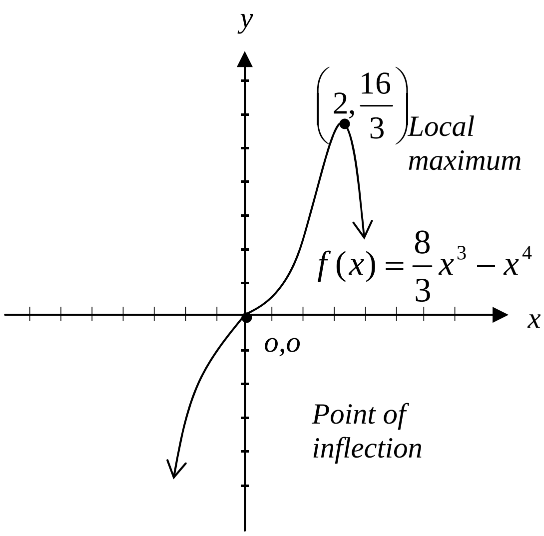
Now, to apply the Second-Derivative Test, we find $f''(0)$ and $f''(2)$.

$f''(x)=16x-12x^2$

$f''(0)=16\times 0-12\times 0^2=0$ the test fails.

$f''(2)=16\times 2-12\times 2^2=-16$, indicating that *f* has a relative maximum at $x=2$. This value is $f(2)=\frac{8}{3}\times 2^3-2^4=\frac{16}{3}$

[The First-Derivative Test will show that *f* increases to the left of x = 0 and to the right of x=0. So x = 0 does not give a local minimum or a local maximum. The test for concavity will show that a point of inflection occurs at x = 0.]



Example 3
Use the second derivative test to find the relative maxima and minima of the function $f(x) = 2x^3 + 3x^2 - 12x - 7$.

(Answer: relative minimum point$(1, -14)$ and relative maximum point$(-2, 13)$)

Example 4
Find the point of diminishing returns for the sales function

$$S(x) = -0.02x^3 + 3x^2 + 100$$

where x represents thousands of dollars spent on advertising, $0 \le x \le 80$ and S is sales in thousands of dollars for automobile tires.

Solution
Find the hypercritical values of x between 0 and 80, and determine whether these points are points of inflection.

$$S(x) = -0.02x^3 + 3x^2 + 100$$
$$S'(x) = -0.06x^2 + 6x$$
$$S''(x) = -0.12x + 6$$

Setting $S''(x) = 0$ gives

$$-0.12x + 6 = 0$$
$$x = 50$$

Testing will show that

$$S''(x) > 0 \text{ for } 0 < x < 50$$
$$S''(x) < 0 \text{ for } 50 < x < 80$$

The **point of diminishing returns** is at $(50, S(50)) = (50, 5100)$, where \$50,000 is spent on advertising, and sales in tires are \$5,100,000.

8 Applications to Business and Economics

Remember that in an **inventory problem** total cost may include **ordering cost** (to cover handling and transportation), **storage cost,** and **purchase cost.** Then we get

$$\text{Total cost} = \text{storage cos} + \text{ordering cost} + \text{purchase cost}$$

Average cost per unit (*AC*) is the total cost divided by the number of units produced. Hence, if $C(q)$ denotes the total cost of producing q units of item, the average cost per unit is $AC(q) = \dfrac{C(q)}{q}$.

We have the relationship between average cost and marginal cost which is stated as follows:

Suppose AC and MC denote the average cost and marginal cost respectively.Then
AC is decreasing when $MC < AC$
AC is increasing when $MC > AC$
AC has (first-order) critical point (usually relative minimum) when $MC = AC$

Students are strongley recommended to do mathematical proof for these facts.

8.1 Elasticity of Demand

A convinience measure of sentitivity of demand to changes in price is the percentage change in demand that is generated by a 1 percent increase in price. If p denotes the price, q the corresponding number of units demanded, and Δp a (small) change in price, the approximation formula for percentage change gives

$$\text{Percentage change in } q \simeq 100\frac{(dq/dp)\Delta p}{q}$$

In particula, if the change in p is a 1-percent increase, then $\Delta p = 0.01p$ and

$$\text{Percentage change in } q \simeq 100\frac{(dq/dp)(0,01p)}{q} = \frac{p}{q}\frac{dq}{dp}$$

The expression on the right-hand side of this approximation is known in economics as the **elasticity demand.** In summary,

> If q denotes the demand for a commodity and p its price, the elasticity of demand, is given by $\eta = \frac{p}{q}\frac{dq}{dp}$. It is the percentage change in demand due to a 1 percentage increase in price.

Example 1

Suppose the demand q and price p for a certain commodity are related by the linear equation $q = 240 - 2p$ (for $0 \le p \le 120$).

a. Express the elasticity of demand as a function of p.
b. Calculate the elasticity of demand when the price is $p = 100$. Inteprete the answer.
c. Calculate the elasticity of demand when the price is $p = 50$. Inteprete the answer.
d. At what price is the elasticity of demand equal to -1?

Solution

a. The elasticity of demand is
$$\eta = \frac{p}{q}\frac{dq}{dp} = \frac{p}{q}(-2) = -\frac{2p}{240-2p} = -\frac{p}{120-p}$$
b. When $p = 100$, the elasticity of demand is
$$\eta = -\frac{p}{120-p} = -\frac{100}{120-100} = -5$$
That is, when the price is $p = 100$, a 1 percent increase in price will produce a decrease in demand of approximately 5 percent.
c. When $p = 50$, the elasticity of demand is
$$\eta = -\frac{p}{120-p} = -\frac{50}{120-50} = -0.71$$
That is, when the price is $p = 50$, a 1 percent increase in price will produce a decrease in demand of approximately 0.71 percent.
d. The elasticity of demand will be equal to $-1$when
$$-1 = -\frac{p}{120-p}$$
solving for p to get $p = 60$.

It means that, at this price ($p = 60$), a one-percent increase in price will result in a decrease in demand of approximately the same percent.

8.2 Levels of Elasicity of Demand

Ingeneral, the elasticity of demand η is negative, since demand decreases as price increases. If $|\eta| > 1$, the percentage decrease in demand is *greater than* the percentage increase in price that caused it. In this situation, economists say that demand is **elastic** with respect to price. If $|\eta| < 1$, the percentage decrease in demand is *less than* the percentage increase in price that caused it. In this situation, economists say that demand is **inelastic** with respect to price. If $|\eta| = 1$ the percentage changes in price and demand are equal, the demand is said to be of **unit elasticity.**

If $|\eta| > 1$, demand is said to be **elastic** with respect to price.

If $|\eta| < 1$, demand is said to be **inelastic** with respect to price.

If $|\eta| = 1$, demand is said to be **unit elasticity** with respect to price.

8.3 Elasticity and the Total Revenue

If *R* denotes the total revenue, *p* the price per unit, and *q* the number of units sold (i.e. the demand), then we can obtain $R = pq$.

The level of the elasticity of demand with respect to price gives useful information about the total revenue obtained from the sale of the product. In particular, if the demand is inelastic ($|\eta| < 1$), the total revenue increases as the price increases (although demand drops). The idea is that, in this case, the relatively small percentage decrease in demand is offset by the larger percentage increase in price, and hence the revenue, which is price times demand, increases. If the demand is elastic ($|\eta| > 1$), the total revenue dereases as the price increases. In this case, the relatively large percentage decrease in demand is not offset by the smaller percentage increase in price. We summary the situation as follows

If demand is inelastic ($|\eta| < 1$), total revenue increases as price increases.

If demand is elastic ($|\eta| > 1$), total revenue decreases as price increases.

(The proof is omitted)

Example 2

Suppose the demand *q* and price *p*for a certain commodity are related by the equation $q = 300 - p^2$ (for $0 \le p \le \sqrt{300}$)

- **a.** Determine where the demand is elastic, inelastic, and of unit elasticity with respect to price.
- **b.** Use the results of part a. to describe the behavior of the total revenue as a function of price.
- **c.** Find the total revenue function explicitly and use its first derivative to determine its intervals of increse and decrease and the price at which revenue is maximized.

Solution

a. The elasticity of demand is

$$\eta = \frac{p}{q}\frac{dq}{dp} = \frac{p}{300-p^2}(-2p) = -\frac{2p^2}{300-p^2}$$

The demand is of unit elasticity when $|\eta| = 1$, that is, when

$$\frac{2p^2}{300-p^2} = 1$$
$$p^2 = 100$$
$$p = \pm 10$$

of which only $p = 10$ is in the relevent interval $0 \le p \le \sqrt{300}$

If $0 \le p < 10$

$$|\eta| = \frac{2p^2}{300-p^2} < \frac{2\times 10^2}{300-10^2} = 1$$

and hence the demand is inelastic.

If $10 < p \le \sqrt{300}$

$$|\eta| = \frac{2p^2}{300-p^2} > \frac{2\times 10^2}{300-10^2} = 1$$

and hence the demand is elastic.

b. The total revenue is an increasing function of p when demand is inelastic, that is, on the interval $0 \le p < 10$ and a decreasing function of p when demand is elastic, that is, on the interval $10 < p \le \sqrt{300}$. At the price $p = 10$ of unit elasticity, the revenue function has a relative maximum.

c. The revenue function is $p = pq$ or

$$R(p) = p\left(300 - p^2\right) = 300p - p^3$$

Its derivative is

$$R'(p) = 300 - 3p^2 = 3(10-p)(10+p)$$

which is zero when $p = \pm 10$, of which only $p = 10$ is in the relevant interval $0 \le p \le \sqrt{300}$.

On the interval $0 \le p < 10$, $R'(p)$ is positive and so $R(p)$ is increasing. On the interval $10 < p \le \sqrt{300}$, $R'(p)$ is negative and so $R(p)$ is decreasing. At the critical value $p = 10$, $R(p)$ stops increasing and starts decreasing and hence has a relative maximum.

Exercises

1 គណនាដេរីវេនៃអនុគមន៍ខាងក្រោម រួចសម្រួលលទ្ធផលឱ្យនៅងាយបំផុត (Differentiate the following function, do as much computation as possible to simplify the results)

a. $y = x^2 + 3x + 3$

b. $f(x) = x^9 + 5x^8 + x + 12$

c. $y = \frac{1}{x} + \frac{1}{x^2} - \frac{1}{\sqrt{x}}$

d. $f(x) = \sqrt{x^3} + \frac{1}{\sqrt{x^3}}$

e. $f(x) = (2x+1)(3x-2)$

f. $f(x) = \left(x^2-5\right)\left(1-2x^2\right)$

g. $f(x) = 100(2x+1)(1-5x)$

h. $y=20\left(4-x^2\right)\left(2x+1\right)$ **i.** $f\left(x\right)=\frac{1}{5}\left(x^3-2x^2+1\right)$

j. $f\left(x\right)=-3\left(5x^3-2x+4\right)$ **k.** $y=\frac{2x-3}{5x+4}$

l. $f\left(x\right)=\frac{3}{x+3}$

2 រកអត្រាប្រែប្រួលនៃអនុគមន៍ $f\left(x\right)$ ធៀបទៅនឹង x ចំពោះតម្លៃ x ដែលគេបញ្ជាក់ (Find the rate of change of the given function $f\left(x\right)$ with respect to x for the prescribed value of x.)

a. $f\left(x\right)=x^3-3x+5,\ x=2$, **b.** $f\left(x\right)=\sqrt{x}+5x, x=4$

c. $f\left(x\right)=\left(x^2+2\right)\left(x+\sqrt{x}\right), x=4$ **d.** $f\left(x\right)=\left(x^2+3\right)\left(5-2x^3\right), x=1$

e. $f\left(x\right)=\frac{2x-1}{3x+5}, x=1$ **f.** $f\left(x\right)=x+\frac{3}{2-4x}, x=0$

3 គេរំពឹងទុកថាក្នុងរយៈពេល t ឆ្នាំគិតពីពេលនេះទៅ ចរាចរណ៍នៃសារពត៌មានក្នុងស្រុកមួយនឹងស្មើទៅនឹង $C\left(t\right)=100t^2+400t+5,000$ ។

ក. ចូរទាញរកកន្សោមមួយដែលតាងឱ្យអត្រាបំរែបំរួលនៃចរាចរណ៍សារពត៌មានធៀបទៅនឹងពេល t ។

ខ. តើចរាចរណ៍សារពត៌មាននឹងប្រែប្រួលក្នុងអត្រាប៉ុន្មានធៀបនឹងពេលក្នុងរយៈពេល៥ឆ្នាំពីពេលនេះ ទៅ? តើនៅពេលនោះចរាចរណ៍សារពត៌មាន កើនឡើង ឬធ្លាក់ចុះ?

គ. តើនៅឆ្នាំទី៦ ចរាចរណ៍សារពត៌មាននឹងប្រែប្រួលជាក់ស្តែងក្នុងទំហំប៉ុន្មាន?

(It is estimated that t years from now, the circulation of a local newspaper will be $C\left(t\right)=100t^2+400t+5,000$. a/. Derive an expression for the rate at which the circulation will be changing with respect to time t years from now. b/. At what rate will the circulation be changing with respect to time 5 years from now? Will the circulation be increasing or decreasing at that time? c/. By how much will the circulation actually change during the 6^{th} year?)

(a/. $C'\left(t\right)=200t+400$,b/.Increasing at the rate of 1,400 per year, c/. 1,500)

4 ការសិក្សាអំពីប្រសិទ្ធភាពការងារវេនព្រឹកនៅក្នុងរោងចក្រមួយបានបង្ហាញឱ្យឃើញថា ជាមធ្យមកម្មករដែលមក ដល់កន្លែងធ្វើការនៅម៉ោង៨:០០ព្រឹក នឹងប្រមូលវិទ្យុបានចំនួន $f\left(x\right)=-x^3+6x^2+15x$ ក្នុងរយៈពេល x ម៉ោង បន្ទាប់ ។

ក. ទាញរកកន្សោមមួយតាងឱ្យអត្រានៃការប្រមូលវិទ្យុរបស់កម្មករបន្ទាប់ពីបានចាប់ផ្តើមការងាររយៈពេល x ម៉ោង ។

ខ. តើអត្រានៃការប្រមូលនៅម៉ោង៩:០០ស្មើនឹងប៉ុន្មាន?

គ. តើកម្មករនោះប្រមូលដាក់ស្តែងបានវិទ្យុប៉ុន្មានគ្រឿងចន្លោះពីម៉ោង ៩:00 ទៅម៉ោង១0:00ព្រឹក?

(An efficiency study of the morning shift at a certain factory indicates that an average worker who arrives on the job at 8:00 A.M. will have assembled $f(x) = -x^3 + 6x^2 + 15x$. a/. Derive a formula for the rate at which the worker will be assembling radios after *x* hours. b/. At what rate will the worker be assembling radios at 9:00 A.M.? C/. How many radios will the worker actually assemble between 9:00 and 10:00 A.M.?)

(a/. $f'(x) = -3x^2 + 12x + 15$, b/. 24 Radios per hour, c/.26)

5 រកអត្រាប្រែប្រួលភាគរយនៃអនុគមន៍ $f(t) = 3t^2 - 7t + 5$ ធៀបទៅនឹង t ពេល$t = 2$ ។ (Find the percentage rate of change in the function $f(t) = 3t^2 - 7t + 5$ with respect to t, when t=2.

6 រកអត្រាប្រែប្រួលភាគរយនៃអនុគមន៍ $f(x) = x(x+3)^2$ ធៀបទៅនឹង x ពេល$x = 3$ ។ (Find the percentage rate of change in the function $f(x) = x(x+3)^2$ with respect to x, when $x = 3$.) (Answer: 66.67 percent)

7 គេទស្សន៍ទាយថាx ខែខាងមុខ អត្រាប្រជាជនរបស់ទីក្រុងមួយស្មើនឹង $P(x) = 2x + 4x^{3/2} + 5,000$ ។

ក. តើអត្រាប្រជាជននឹងប្រែប្រួលក្នុងអត្រាប៉ុន្មានក្នុងពេល៩ខែខាងមុខ?

ខ. តើអត្រាប្រជាជននឹងប្រែប្រួលក្នុងអត្រាប៉ុន្មាន**ភាគរយ** ក្នុងពេល៩ខែខាងមុខ?

(It is projected that *x* months from now, the population of a certain town will be $P(x) = 2x + 4x^{3/2} + 5,000$. a/. At what rate will the population be changing with respect to time 9 months from now? b/. At what percentage rate will the population be changing with respect to time 9 months from now?)

8 ប្រាក់ចំណូលប្រចាំឆ្នាំសរុបរបស់ក្រុមហ៊ុនមួយ គឺ $A(t) = 0.1t^2 + 10t + 20$ ពាន់ដុល្លា រយៈពេល t ឆ្នាំបន្ទាប់ពី ការបង្កើតនៅក្នុងឆ្នាំ១៩៨៧ ។

ក. តើចំណូលប្រចាំឆ្នាំសរុបរបស់ក្រុមហ៊ុន នឹងកើនក្នុងអត្រាប៉ុន្មានធៀបទៅនឹងពេលនៅក្នុងឆ្នាំ១៩៩១?

ខ. តើចំណូលប្រចាំឆ្នាំសរុបរបស់ក្រុមហ៊ុន នឹងកើនក្នុងអត្រាប៉ុន្មាន**ភាគរយ**ធៀបទៅនឹងពេល នៅក្នុងឆ្នាំ១៩៩១?

(The gross annual earnings of a certain company were $A(t) = 0.1t^2 + 10t + 20$ thousand dollars *t* years after its formation in 1987. a/ At what rate were the gross annual earnings of the company growing with respect to time in 1991? b/. At what percentage rate were the gross annual earnings of the company growing with respect to time in 1991?) (a/. 10,800 people, b/.17.53 percent).

9 តាមរយៈកំណត់ត្រាបានបង្ហាញថា ក្នុងរយៈពេល x ឆ្នាំ បន្ទាប់ពីឆ្នាំ១៩៨៥ទៅ **ពន្ធលើកម្មសិទ្ធិ** ជាមធ្យមចំពោះផ្ទះ ដែលមានបន្ទប់៣ នៅក្នុងសហគមន៍មួយកំណត់ដោយអនុគមន៍

$T(x)=20x^2+40x+600$ ។

ក. តើពន្ធលើកម្មសិទ្ធិកើនក្នុងអត្រាប៉ុន្មានធៀបទៅនឹងពេល នៅក្នុងឆ្នាំ១៩៩១?

ខ. តើពន្ធលើកម្មសិទ្ធិ កើនក្នុងអត្រា**ប៉ុន្មានភាគរយ** ធៀបទៅនឹងពេល នៅក្នុងឆ្នាំ១៩៩១?

(Records indicate that x years after 1985, the average property tax on a three-bedroom home in a certain community was $T(x)=20x^2+40x+600$. a/. At what rate was the property tax increasing with respect to time in 1991? b/. At what percentage rate was the property tax increasing with respect to time in 1991?)

10 គេប៉ាន់ស្មានថា t ឆ្នាំទៅមុខទៀតអត្រាប្រជាជននៅក្នុងទីក្រុងមួយនឹងស្មើទៅនឹង

$P(t)=t^2+200t+10,000$ ។

ក. សរសេរកន្សោមមួយតាងឱ្យអត្រាបំរែបំរួលភាគរយនៃអត្រាប្រជាជនជាអនុគមន៍នៃ t រួចសង់ក្រាហ្វ ។

ខ. តើនៅ ក្នុងរយៈពេលយូរទៅខាងមុខទៀតអត្រាបំរែបំរួលភាគរយនេះទៅជាយ៉ាងណា?

(It is estimated that t years from now, the population of a certain town will be $P(t)=t^2+200t+10,000$. a/. Express the percentage rate of change of the population as a function of t, simplify this function algebraically, and draw it graph. b/. What will happen to the percentage rate of change of the population in the long run?) (a/. $\frac{20}{t+100}$ percent, b/. 0)

11 ផលទុនជាតិសរុបរបស់ប្រទេសមួយកើនឡើងក្នុងអត្រាថេរ ។ នៅក្នុងឆ្នាំ១៩៨៦ ផលទុនជាតិសរុបស្មើនឹង ១២៥ ពាន់លានដុល្លា ហើយនៅឆ្នាំ១៩៨៨ ផលទុនជាតិសរុបស្មើនឹង១៥៥ពាន់លានដុល្លា ។ តើនៅក្នុងឆ្នាំ១៩៩១ ផលទុនជាតិ សរុបនឹងកើនឡើងនៅក្នុងអត្រាប៉ុន្មានភាគរយ? (The gross national product (GNP) of a certain country is growing at a constant rate. In 1986 the GNP was 125 billion dollars, and in 1988 the GNP was 155 billion dollars. At what percentage rate was the GNP growing in 1991?) (Answer: 7.5 percent)

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12 ចូររកតម្លៃប្រហែលនៃទំហំប្រែប្រួលរបស់អនុគមន៍ $f(x)=x^2-3x+5$ បើ x កើនពី៥ ទៅ ៥.៣ ។ (Estimate how much the function $f(x)=x^2-3x+5$ will change as x increases from 5 to 5.3.) (Answer: 2.1)

13 ចូររកតម្លៃប្រហែលនៃទំហំប្រែប្រួលរបស់អនុគមន៍ $f(x)=\frac{x}{x+1}-3$ បើ x ថយចុះពី4 ទៅ3.8 ។

(Estimate how much the function $f(x)=\frac{x}{x+1}-3$ will change as x decreases from 4 to 3.8)

14 ចូររកតម្លៃប្រហែលនៃបំរែបំរួលជាភាគរយរបស់អនុគមន៍ $f(x)=x^2+2x-9$ ពេល x កើនពី4 ទៅ4.3 ។ (Estimate the percentage change in the function $f(x)=x^2+2x-9$ as x increases from 4 to 4.3.)

15 ចូររកតម្លៃប្រហែលនៃបំរែបំរួលជាភាគរយរបស់អនុគមន៍ $f(x)=3x+\frac{2}{x}$ ពេល x ថយចុះពី5 ទៅ 4.6 ។ (Estimate the percentage change in the function $f(x)=3x+\frac{2}{x}$ as x decreases from 5 to 4.6.)

16 ថ្លៃដើមសរុបរបស់ផលិតករម្នាក់ គឺ $C(q)=0.1q^3-0.5q^2+500q+200$ ដុល្លានៅពេលដែលកំរិតផលិតកម្មស្មើនឹង q ឯកតា ។ កំរិតចរន្តរបស់ផលិតកម្មគឺ៤ឯកតាហើយផលិតករនោះគ្រោងនឹងបង្កើនឱ្យបាន៤.១ឯកតា ។ ប៉ាន់ស្មាន ឬរកតម្លៃប្រហែលនៃទំហំបំរែបំរួលរបស់ថ្លៃដើមសរុប ។ (A manufacturer's total cost is $C(q)=0.1q^3-0.5q^2+500q+200$ dollars when the level of production is q units. The current level of production is 4 units, and the manufacturer is planning to increase this to 4.1 units. Estimate how the total cost will change as a result.) (Answer: $50.08)

17 គេរំពឹងទុកថា t ឆ្នាំទៅមុខទៀត ចរាចរណ៍នៃសារពតិមានក្នុងស្រុកមួយកំណត់ដោយ

$$C(t)=100t^2+400t+5,000 \text{ ។}$$

រកតម្លៃប្រហែលនៃបរិមាណកំណើនចរាចរណ៍សារពតិមានក្នុងកំលុង៦ ខែបន្ទាប់ ។ ណែនាំ: តម្លៃចរន្តនៃអថេរគឺ $t=0$ ។ (It is projected that t years from now, the circulation of a local newspaper will be $C(t)=100t^2+400t+5,000$. Estimate the amount by which the circulation will increase during the next 16 months. (Hint: the current value of the variable is $t=0$). (Answer: 200)

18 ការសិក្សាលើបរិស្ថាននៅក្នុងសហគមន៍មួយបានលើឡើងថា រយៈពេល t ឆ្នាំពីពេលនេះទៅមុខកំរិតជាមធ្យមនៃ កាបូនម៉ូណូអុកស៊ីតនៅក្នុងខ្យល់ស្មើនឹង $Q(t)=0.05t^2+0.1t+3.4$ ភាគក្នុងមួយលានភាគ ។ តើនៅក្នុងកំលុង៦ ខែខាងមុខ កំរិតកាបូនម៉ូណូអុកស៊ីតនឹងប្រែប្រួលប្រហែលប៉ុន្មាន? (An environmental study of a certain community suggests that t years from now, the average level of carbon monoxide in the air will be $Q(t)=0.05t^2+0.1t+3.4$ parts per million. By approximately how much will the carbon monoxide level change during the coming 6 months?) (Answer: 0.05 parts per million)

19 ការសិក្សាលើប្រសិទ្ធភាពការងារនៃវេនពេលព្រឹកនៅក្នុងរោងចក្រមួយបានបញ្ជាក់ថា ជាមធ្យមកម្មករដែលមកដល់កន្លែងធ្វើការនៅម៉ោង៨:០០នឹងប្រមូលវិទ្យុបាន $f(x)=-x^3+6x^2+15x$ ក្នុងរយៈពេល x ម៉ោងបន្ទាប់ ។ តើកម្មករនោះប្រមូលវិទ្យុបានប្រហែលប៉ុន្មានក្នុងចន្លោះពីម៉ោង

៩:00 និង ៩:១៥? (An efficiency study of the morning shift at a certain factory indicates that an average worker who arrives on the job at 8:00 A.M. will have assembled $f(x) = -x^3 + 6x^2 + 15x$ radios x hours later. Approximately how many radios with the worker assemble between 9:00 and 9:15 A.M?)

20 នៅក្នុងរោងចក្រមួយ ផលិតផលប្រចាំថ្ងៃស្មើនឹង $Q(k) = 600k^{1/2}$ ឯកតា ដែល k តាងឱ្យ **វិនិយោគមូលធន** គិតជាឯកតា១000ដុល្លា ។ **វិនិយោគមូលធនចរន្ត** ស្មើនឹង ៩00000ដុល្លា ។ ចូរប៉ាន់ស្មានអំពីឥទ្ធិពលដែលវិនិយោគមូលធនបន្ថែម ចំនួន៨00ដុល្លា នឹងមានទៅលើផលិតផលប្រចាំថ្ងៃ ។ (At a certain factory, the daily output is $Q(k) = 600k^{1/2}$ units, where k denotes the capital investment measured in units of \$ 1,000. The current capital investment is \$900,000. Estimate the effect that an additional capital investment of \$800 will have on the daily output.) (Answer: 8 units)

21 នៅក្នុងរោងចក្រមួយ ផលិតផលប្រចាំថ្ងៃគឺ $Q(L) = 60,000L^{1/3}$ ឯកតា ដែល L តាងឱ្យទំហំកំលាំងពលកម្ម គិតជា **ម៉ោងកម្មករ** ។ ថ្មីៗនេះ កំលាំងពលកម្មចំនួន១000ម៉ោងកម្មករត្រូវបានប្រើប្រាស់ជារៀងរាល់ថ្ងៃ ។ ចូរប៉ាន់ស្មាន ពីឥទ្ធិពលទៅលើផលិតកម្មដែលកើតឡើង បើកំលាំងពលកម្មត្រូវកាត់បន្ថយមកនៅ ៩៤0ម៉ោងកម្មករ. (At a certain factory, the daily output is $Q(L) = 60,000L^{1/3}$ units, where L denotes the size of the labor force measured in worker-hours. Currently 1000 worker-hours of labor are use each day. Estimate the effect on output that will be produced if the labor force is cut to 940 worker-hours.)

22 នៅក្នុងរោងចក្រមួយ ផលិតផលប្រចាំថ្ងៃកំណត់ដោយអនុគមន៍ $Q = 3,000K^{1/2}L^{1/3}$ ឯកតា ដែល k តាងឱ្យវិនិយោគមូលធនរបស់ក្រុមហ៊ុនដែលគិតជាឯកតា១000ដុល្លា ហើយ L តាងឱ្យទំហំកំលាំងពលកម្មគិតជាម៉ោងកម្មករ ។ ឧបមាថាវិនិយោគមូលធនចរន្តគឺ៤00000ដុល្លា ហើយថាកំលាំងពលកម្មចំនួន ១៣៣១ ម៉ោងកម្មករ ត្រូវបានប្រើប្រាស់រាល់ថ្ងៃ ។ ចូរប្រើការវិភាគម៉ាជីណលដើម្បីប៉ាន់ស្មានអំពីឥទ្ធិពលដែល វិនិយោគមូលធនបន្ថែម១000ដុល្លា នឹងមានទៅលើផលិតផលប្រចាំថ្ងៃ បើទំហំកំលាំពលកម្មពុំផ្លាស់ប្តូរ ។ (At a certain factory, the daily output is $Q = 3,000K^{1/2}L^{1/3}$ where K denotes the firm's capital investment measured in units of \$1,000 and L denotes the size of the labor force measured in worker-hours. Suppose that the current capital investment is \$400,000 and that 1,331 worker-hours of labor are used each day. Use marginal analysis to estimate the effect that an additional capital investment of \$1,000 will have on the daily output if the size of the labor force is not changed.)(Answer: 825).

23 ផលិតផលប្រចាំថ្ងៃនៅក្នុងរោងចក្រមួយកំណត់ដោយ $Q(L) = 300L^{2/3}$ ឯកតា ដែល L តាងឱ្យទំហំកំលាំងពលកម្មដែលគិតជាម៉ោងកម្មករ ។ បច្ចុប្បន្ននេះ កំលាំងពលកម្ម៥១២ម៉ោងត្រូវបាន

ប្រើប្រាស់រាល់ថ្ងៃ ។ ចូរប៉ាន់ស្មានទូវ កំលាំងពលកម្មបន្ថែមដែលគិតជាម៉ោងដើម្បីបង្កើនផលិតផលប្រចាំថ្ងៃក្នុងកំរិត12.5 ឯកតា ។ (The daily output at a certain factory is $Q(L)=300L^{2/3}$ units, where L denotes the size of the labor force measured in worker-hours. Currently 512 worker-hours of labor are used each day. Estimate the number or additional worker-hours of labor that will be needed to increase daily output by 12.5 units.) (Answer: 0.5)

24 ថ្លៃដើមរបស់ផលិតករម្នាក់កំណត់ដោយអនុគមន៍ $C(q)=\frac{1}{6}q^3+642q+400$ នៅពេលដែល q ឯកតាត្រូវបានផលិត ។ កំរិតចរន្ត(ធម្មតា) នៃផលិតកម្មស្មើនឹង4 ឯកតា ។ ចូរប៉ាន់ស្មានទូវបរិមាណផលិតដែលផលិតករគួរកាត់បន្ថយ ដើម្បីកាត់បន្ថយថ្លៃដើម 130$ ។ (A manufacturer's total cost is $C(q)=\frac{1}{6}q^3+642q+400$ dollars when q units are produced. The current level of production is 4 units. Estimate the amount by which the manufacturer should decrease production to reduce the total cost by $130.)

25 តាមកំណត់ត្រាបានបញ្ជាក់ថា x ឆ្នាំបន្ទាប់ពីឆ្នាំ១៩៨៨ ពន្ធលើកម្មសិទ្ធជាមធ្យម ទៅលើផ្ទះដែលមានបន្ទប់បី នៅក្នុងសហគមន៍មួយកំណត់ដោយ $T(x)=60x^{3/2}+40+1,200$ ដុល្លា ។ ប៉ាន់ស្មានទូវកំណើនពន្ធជាភាគរយក្នុងកំលុងឆមាសទី១នៃឆ្នាំ១៩៩២ ។(Records indicate that x years after 1988, the average property tax on a three-bedroom home in a certain community was $T(x)=60x^{3/2}+40x+1,200$ dollars. Estimate the percentage by which the property tax increased during the first half of 1992.) (Answer: 6)

26 ផលិតផលនៅក្នុងរោងចក្រមួយគឺ $Q(K)=400K^{1/2}$ ឯកតា ដែល K តាងឱ្យមូលធនវិនិយោគរបស់ក្រុមហ៊ុន ។ ប៉ាន់ស្មានទូវកំណើនជាភាគរយនៃផលិតផល ដែលជាលទ្ធផលនៃការកើនឡើង ១ភាគរយរបស់មូលធនវិនិយោគ ។ (The output at a certain factory is $Q(K)=400K^{1/2}$ units, where K denotes the firm's capital investment. Estimate the percentage increase in output that will result from a 1 percent increase in capital investment.)(Answer: 0.5 percent)

27 ផលិតផលចេញពីរោងចក្រមួយកំណត់ដោយអនុគមន៍ $Q=600K^{1/2}L^{1/3}$ ឯកតា ដែល K តាងឱ្យមូលធនវិនិយោគ ហើយ L តាងឱ្យទំហំកំលាំងពលកម្ម ។ចូរប៉ាន់ស្មានកំណើនជាភាគរយនៃផលិតផលដែលកើតឡើងដោយសារកំណើន២ភាគរយរបស់កំលាំងពលកម្មបើវិនិយោគមូលធនពុំ ប្រែប្រួល ។ (The output at a certain factory is $Q=600K^{1/2}L^{1/3}$ where K denotes the capital investment and L the size of the labor force. Estimate the percentage increase in output that will result from 2 percent increase in the size of the labor force if capital investment is not changed.) (Answer: 0.67 percent)

28 នៅក្នុងរោងចក្រមួយ ផលិតផលប្រចាំថ្ងៃគឺ $Q(K)=1,200K^{1/2}$ ឯកតា ដែល K តាងឱ្យវិនិយោគមូលធនរបស់ក្រុមហ៊ុន ។ ប៉ាន់ស្មានកំណើនជាភាគរយនៃវិនិយោគមូលធនដែលត្រូវការ ដើម្បីបង្កើនផលិតផល 1.2 ភាគរយ ។ (At a certain factory, the daily output is $Q(K)=1,200K^{1/2}$ units, where K denotes the firm's capital investment. Estimate the percentage increase in capital investment that is needed to produce a 1.2 percent increase in output.)

29 ឧបមាថាថ្លៃដើមសរុបគិតជាដុល្លាលើការផលិត q ឯកតាគឺ $C(q)=3q^2+q+500$ ។

ក. ប្រើការវិភាគ ម៉ាជីណល ដើម្បីប៉ាន់ស្មានថ្លៃដើមលើការផលិតឯកតាទី41 ។

ខ. គណនាថ្លៃដើមជាក់ស្តែងលើការផលិតឯកតាទី41 ។

(Suppose the total cost in dollars of manufacturing q units is $C(q)=3q^2+q+500$. a/. Use marginal analysis to estimate the cost of manufacturing the 41th unit. b/. Compute the actual cost of manufacturing the 41th.)

30 ថ្លៃដើមសរុបរបស់ផលិតករម្នាក់កំណត់ដោយ $C(q)=0.1q^3-0.5q^2+500q+200$ ដុល្លា ដែល q ជាចំនួន ឯកតាដែលត្រូវផលិត ។

ក. ប្រើការវិភាគម៉ាជីណល ដើម្បីប៉ាន់ស្មានថ្លៃដើម្បីលើការផលិតឯកតាទី ៤ ។

ខ. គណនាថ្លៃ ដើមជាក់ស្តែងលើការផលិតឯកតាទី៤ ។

(A manufacturer's total cost is $C(q)=0.1q^3-0.5q^2+500q+200$ dollars, where q is the number of units produced. a/. Use marginal analysis to estimate the cost of manufacturing the 4th unit. b/. Compute the actual cost of manufacturing the 4th unit.)

31 Use the chain rule to compute the derivative $\frac{dy}{dx}$ and simplify your answer.

a. $y=u^2+1, u=3x-2$ b. $y=2u^2-u+5, u=1-x^2$

c. $y=\sqrt{u}, u=x^2+2x-3$ d. $y=\frac{1}{u^2}, u=x^2+1$

e. $y=\frac{1}{\sqrt{u}}, u=x^2-9$ f. $y=u^2, u=\frac{1}{x-1}$

32 Differentiate the following function

a/. $f(x)=\frac{1}{\sqrt{4x^2+1}}$ b/. $f(x)=\frac{3}{(1-x^2)^4}$ c/. $f(x)=\left(1+\sqrt{3x}\right)^5$

33 ប្រាក់ចំណូលប្រចាំឆ្នាំសរុបនៃក្រុមហ៊ុនមួយគឺ $f(t)=\sqrt{10t^2+t+236}$ ពាន់ដុល្លា ក្នុងរយៈពេល t ឆ្នាំ ក្រោយពី ការបង្កើតនៅក្នុងខែមករាឆ្នាំ1988 ។

ក. តើនៅក្នុងខែមករាឆ្នាំ 1992 ប្រាក់ចំណូលសរុប របស់ក្រុមហ៊ុននឹងកើនក្នុង **អត្រាប៉ុន្មាន**?

ខ. តើនៅក្នុងខែមករាឆ្នាំ 1992 ប្រាក់ចំណូលសរុប របស់ក្រុមហ៊ុននឹងកើនក្នុង**អត្រាប៉ុន្មានភាគរយ**?

(The gross annual earnings of a certain company were $f\left(t\right)=\sqrt{10t^2+t+236}$ thousand dollars t years after its formation in January 1988. a/. At what rate were the gross annual earnings of the company growing in January 1992? b/. At what percentage rate were the gross annual earnings growing in January 1992? (a/. \$2,025 per year, b/.10.125 percent per year.)

34 ការសិក្សាលើប្រសិទ្ធិភាពការងារវេនពេលព្រឹកនៅក្នុងរោងចក្រមួយបានបង្ហាញឱ្យឃើញថាជាមធ្យមកម្មករដែលមកដល់កន្លែងធ្វើការងារនៅម៉ោង៨:00 នឹងផលិតបាន

$$Q\left(t\right)=-t^3+8t^2+15t$$

ឯកតានៅ t ម៉ោងបន្ទាប់ ។

ក. គណនាអត្រាផលិតកម្មនៅរបស់កម្មករនៅម៉ោង ៩:00ព្រឹក ។

ខ. តើក្នុងអត្រាប៉ុន្មានដែលអត្រាផលិតកម្មរបស់កម្មករប្រែប្រួលធៀបទៅនឹងពេលនៅម៉ោង៩:00ព្រឹក ។

គ. ប៉ាន់ស្មានទំហំបំរែបំរួលនៃអត្រាផលិតកម្មរបស់កម្មករនៅក្នុងចន្លោះពីម៉ោង៩:00 ទៅម៉ោង៩:១៥ ។

ឃ.គណនាទំហំបំរែបំរួលជាក់ស្តែងនៃអត្រាផលិតកម្មរបស់កម្មករនៅក្នុងចន្លោះពីម៉ោង៩:00 ទៅម៉ោង ៩:១៥ ។

(An efficiency study of the morning shift at a certain factory indicates that an average worker who arrives on the job at 8:00AM. Will have produced $Q(t)=-t^3+8t^2+15t$ units t hours later. a/. Compute the worker's rate of production at 9:00A.M. b/. At what rate is the worker's rate of production changing with respect to time at 9:00A.M? c/.Use calculus to estimate the change in the worker's rate of production between 9:00 and 9:15A.M. d/. Compute the actual change in the worker's rate of production between 9:00 and 9:15A.M.)

35 Suppose the total cost in dollars of manufacturing q units of a certain commodity is $C\left(q\right)=3q^2+5q+75$.

a. At what level of production is the average cost per unit the smallest?
b. At what level of production is the average cost per unit equal to the marginal cost?

ឧបមាថាថ្លៃដើមគិតជាដុល្លាលើការផលិត q ឯកតានៃមុខទំនិញមួយប្រភេទគឺ $C\left(q\right)=3q^2+5q+75$ ។

ក. តើកំរិតផលិតកម្ម q ស្មើប៉ុន្មានដែលធ្វើឱ្យអនុគមន៍ចំណាយមធ្យមក្នុង១ឯកតា តូចបំផុត?

ខ. តើកំរិតផលិតកម្ម q ស្មើប៉ុន្មានទើបអនុគមន៍ចំណាយមធ្យមក្នុង១ឯកតាស្មើនឹងអនុគមន៍ចំណាយម៉ាជីន?

36 The problem is the same as that in problem **35** for $C(q)=q^3+5q+162$.ធ្វើដូច លំហាត់ទី**35**ចំពោះអនុគមន៏$C(q)=q^3+5q+162$ ។

37 Suppose the total revenue in dollars from the sale of *q* units of a certain commodity is $R(q)=-2q^2+68q-128$

a. At what level of sales is the average revenue per unit equal to the marginal revenue?
b. Verify that the average revenue is increasing if the level of sales is less than the level in part a. and decreasing if the level of sales is greater than the level in part **a**.

ឧបមាថាចំណូលសរុបគិតជាដុល្លាពីការលក់ទំនិញមួយប្រភេទចំនួនqឯកតាកំណត់ដោយ $R(q)=-2q^2+68q-128$ ។ **ក.** តើបរិមាណនៃការលក់ស្មើនឹងប៉ុន្មានឯកតា ទើបអនុគមន៏ចំណូលក្នុង១ឯកតាស្មើទៅនឹងចំណូលម៉ាជីន? **ខ.**ចូរផ្ទៀងផ្ទាត់ថាអនុគមន៏ចំណូលមធ្យមកើនបើកំរិតនៃការលក់តូចជាងកំរិតដែលបានរកឃើញក្នុងសំណួរ **ក.** ហើយថាអនុគមន៏ចំណូលមធ្យមចុះ បើ កំរិតនៃការលក់ធំ ជាងកំរិតដែលបានរកឃើញក្នុងសំណួរ **ក.** ។

38 Assume that total national consumption is given by a function $C(x)$ where *x* is the total national income. The derivative $C'(x)$ is called the **marginal propensity to consume,** and if $S=x-C$ represents total national savings, then $S'(x)$ is called the **marginal propensity to save.** Suppose the consumption function is $C(x)=8+0.8x+0.8\sqrt{x}$. Find the marginal prpensity to consume and determine the value of *x* that results in the smallest total savings.

39 Suppose that the demand equation for a certain commodity is $q=60-0.1p$ (for $0\le p\le 600$).

a. Express the elasticity of demand as a function of *p*.
b. Calculate the elasticity of demand when the price is $p=200$. Interpret the answer.
c. At what price is the elasticity of demand equal to -1?

ឧបមាថាសាមីការតម្រូវការមុខទំនិញមួយប្រភេទកំណត់ដោយ$q=60-0.1p$ (ចំពោះ $0\le p\le 600$) ។ **ក.** ចូរសរសេរកន្សោមអេឡាស្ទិចនៃតម្រូវការជាអនុគមន៏នឹងp ។ **ខ.**គណនាអេឡាស្ទិចនៃតម្រូវការនៅពេលដែលថ្លៃ $p=200$ រួចបកស្រាយចម្លើយ ។ **គ.** តើថ្លៃ ស្មើប៉ុន្មានទើបអេឡាស្ទិចនៃតម្រូវការស្មើ-1?

40 Suppose that the demand equation for a certain commodity is $q=200-2p^2$ (for $0\le p\le 10$).

a. Express the elasticity of demand as a function of *p*.
b. Calculate the elasticity of demand when the price is $p=6$. Interpret the answer.
c. At what price is the elasticity of demand equal to -1?

ឧបមាថាសមីការតម្រូវការនៃមុខទំនិញមួយប្រភេទកំណត់ដោយ $q = 200 - 2p^2$ (ចំពោះ $0 \le p \le 10$) ។

ក. ចូរសរសេរកន្សោមអេឡាស្ទិចនៃតម្រូវការជាអនុគមន៍នឹង p ។ **ខ.** គណនាអេឡាស្ទិចនៃតម្រូវការ ពេល $p = 6$ រួចបកស្រាយលទ្ធផល ។ **គ.** តើថ្លៃស្មើប៉ុន្មានទើបអេឡាស្ទិចនៃតម្រូវការ ស្មើទៅនឹង -1?

41 Suppose that the demand equation for a certain commodity is $q = 500 - 2p$ (for $0 \le p \le 250$).

- **a.** Determine where the demand is elastic, inelastic, and of unit elasticity with respect to price.
- **b.** Use the results from part a. to determine the intervals of increase and decrease of the revenue function and the price at which revenue is maximized.
- **c.** Find the total revenue function explicitly and use its first derivative to determine its intervals of increase and decrease and price at which revenue is maximized.

ឧបមាថាសមីការតម្រូវការនៃមុខទំនិញមួយប្រភេទកំណត់ដោយ $q = 500 - 2p$ (ចំពោះ $0 \le p \le 250$) ។

ក.ចូររកចន្លោះដែលនៅក្នុងនោះ តម្រូវការមានភាពអេឡាស្ទិច មិនអេឡាស្ទិច និងអេឡាស្ទិចឯកតាធៀបទៅនឹងថ្លៃ ។ **ខ.**ប្រើលទ្ធផលរបស់សំណួរ**ក.** ចូររកចន្លោះដែលអនុគមន៍ប្រាក់ចំណូលកើនឡើង និងថយចុះ ព្រមទាំថ្លៃ ដែលធ្វើឱ្យអនុគមន៍ប្រាក់ចំណូលមានតម្លៃអតិបរមា ។ **គ.** ចូររកអនុគន៏ប្រាក់ចំណូលដាច់ដោយឡែក រួចប្រើដេរីវេទី១របស់វាដើម្បីរកចន្លោះដែលវាកើន និងចុះ ព្រមទាំងថ្លៃដែលធ្វើឱ្យអនុគមន៍មានអតិបមា ។

42 Suppose that the demand equation for a certain commodity is $q = 120 - 0.1p^2$ for ($0 \le p \le \sqrt{1,200}$)

- a. Determine where the demand is elastic, inelastic, and of unit elasticity with respect to price.
- b. Use the results of part a. to determine the intervals of increase and decrease of the revenue function and the price at which revenue is maximized.
- c. Find the total revenue function explicitly and use its first derivative to determine its intervals of increase and decrease and the price at which revenue is maximized. ឧបមាថាសមីការតម្រូវការនៃមុខទំនិញមួយប្រភេទកំណត់ដោយ $q = 120 - 0.1p^2$ (ចំពោះ $0 \le p \le \sqrt{1,200}$) ។

ក.ចូររកចន្លោះដែលនៅក្នុងនោះ តម្រូវការមានភាពអេឡាស្ទិច មិនអេឡាស្ទិច និងអេឡាស្ទិចឯកតាធៀបទៅនឹងថ្លៃ ។ **ខ.**ប្រើលទ្ធផលរបស់សំណួរ**ក.** ចូររកចន្លោះដែលអនុគមន៍ប្រាក់ចំណូលកើនឡើង និងថយចុះ ព្រមទាំថ្លៃ ដែលធ្វើឱ្យអនុគមន៍ប្រាក់ចំណូលមានតម្លៃអតិបរមា ។ **គ.** ចូររកអនុគន៏ប្រាក់ចំណូលដាច់ដោយឡែក រួចប្រើដេរីវេទី១របស់វាដើម្បីរកចន្លោះដែលវាកើន និងចុះ ព្រមទាំងថ្លៃដែលធ្វើឱ្យអនុគមន៍មានអតិបមា ។

43 Suppose the demand q and price p for a certain commodity are related by the equation $p = 60 - 2q$ for ($0 \le q \le 30$)

- **a.** Express the elasticity of demand as a function of q.

b. Calculate the elasticity of demand when $q = 10$. Interpret the answer.

c. Substitute for q in the formula in part a. to express the elasticity of demand as a function of *p*.

d. Use our original definition of η to express the elasticity of demand as a function of *p*. ឧបមាថា តម្រូវការ q និងថ្លៃ p នៃមុខទំនិញមួយប្រភេទទាក់ទងគ្នាតាមរយៈសមីការ $p = 60 - 2q$ (ចំពោះ $0 \le q \le 30$) ។ **ក.** ចូរសរសេរអេឡាស្ទិចនៃតម្រូវការជាអនុគមន៍ទៅនឹង p ។ **ខ.** គណនាអេឡាស្ទិចនៃតម្រូវការនៅពេលដែល $q = 10$ រួចបកស្រាយចម្លើយ ។ **គ.** ដោយការជំនួសqនៅក្នុងរូបមន្ត សំណួរ**ក.** ចូររកកន្សោមអេឡាស្ទិចនៃតម្រូវការជាអនុគមន៍ទៅនឹងp ។ **ឃ.** ចូរប្រើនិយមន័យដើមរបស់ η ដើម្បីសរសេរកន្សោមអេឡាស្ទិចនៃតម្រូវការជាអនុគមន៍នៃ p ។

Chapter 3

Functions of Two Variables

1 Functions of Two Variables

A function f of the two variables x and y is a rule that assigns to each ordered pair (x, y) of real numbers in some set one and only one real number denoted by $f(x, y)$.

The Domain of a Function of Two Variables
The domain of the function $f(x, y)$ is the set of all ordered pairs (x, y) of real numbers for which $f(x, y)$ can be evaluated.

Example 1
For $f(x, y) = 3x + y^2$. Find **a.** $f(2,3)$, **b.** $f(2, \sqrt{2})$?

Solution

a. $f(2,3) = 3\times 2 + 3^2 = 15$

b. $f(2, \sqrt{2}) = 3\times 2 + (\sqrt{2})^2 = 8$

Example 2
For $f(x, y) = e^{\sqrt{x}} + \ln y$. Find **a.** the domain of *f*. **b.** $f(0,1)$

Solution

a. For $\sqrt{x}$ to be defined, we must have $x \geq 0$. For ln*y* to be defined, we must have *y>0*. So the domain is $\{(x, y) / x \geq 0 \text{ and } y > 0\}$

b. $f(0,1) = e^{\sqrt{0}} + \ln 1 = e^0 + 0 = 1$

Example 3
Suppose $f(x, y) = \dfrac{3x^2 + 5y}{x - y}$. a. Find the domain of *f*. b. Compute $f(1, -2)$.

Example 4
A pharmacy sells two brands of aspirin. Brand A sells for \$1.25 per bottle and Brand B sells for \$1.50 per bottle.

a. What is the revenue function for aspirin?

b. What is the revenue for aspirin if 100 bottles of Brand A and 150 bottles B are sold?

Solution

a. Let x= the number of bottles of Brand A sold and y= the number of bottles of Brand B sold.Then, the revenue function is

$$R(x, y) = 1.25x + 1.50y$$

b. $R(100,150) = 1.25\times 100 + 1.50\times 150 = 125 + 225 = 350$

Economists use a formula called the **Cobb-Douglas Production Functions** to model the production levels of a company (or a country). Output Q at a factory is often regarded as a function of the amount K of capital investment and the size L of the labor force. Output functions of the form

$$Q(K,L) = AK^{\alpha}L^{1-\alpha}$$

where A and α are positive constants and $0<\alpha<1$ have proved to be especially useful in economic analysis. Such functions are known as **Cobb-Douglas production function.**

Example 5

Suppose that the function $Q(x,y) = 500x^{0.3}y^{0.7}$ represents the number of units produced by a company with x units of labor and y units of capital.

- **a.** How many units of a product will be manufactured if 300 units of labor and 50 units of capital are used?
- **b.** How many units will be produced if twice the number of units of labor and capital are used?

Solution

a. $Q(300,50) = 500(300)^{0.3}(50)^{0.7} = 500\times 5.535\times 15.462 = 42,791$ units

b. If number of units of labor and capital are both doubled, then
$x = 2\times 300 = 600$ and $y = 2\times 50 = 100$

$Q(600,100) = 500(600)^{0.3}(100)^{0.7} = 500\times 6.815\times 25.119 = 85,592$ units

Thus we see that production is doubled if both labor and capital are doubled.

2 Partial Derivatives

Definition

Let $z = f(x,y)$

a. The first parital derivative of f with respect to x is:

$$\frac{\partial z}{\partial x} = f_x(x,y) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

b. The first partial derivative of f with respect to y is:

$$\frac{\partial z}{\partial y} = f_y(x,y) = \lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

2.1 Computation of Partial Derivatives

The function $\frac{\partial z}{\partial x}$ or f_x is obtained by differentiating f with respect to x, treating y as a constant.

The function $\frac{\partial z}{\partial y}$ or f_y is obtained by differentiating f with respect to y, treating x as a constant.

Example 1

For the function $f(x,y) = 4x^2 - 3xy + 5y^2$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$?

Solution

Treating y as a constant, we obtain

$$\frac{\partial z}{\partial x} = 8x - 3y$$

Treating x as a constant, we obtain

$$\frac{\partial z}{\partial y} = -3x + 10y$$

Example 2

Find the partial derivatives f_x and f_y if $f(x,y) = x^2 + 2xy^2 + \dfrac{2y}{3x}$.

Example 3

For the function $f(x,y) = xe^{xy} + y^2$, find $f_x(1,2)$ and $f_y(1,2)$.

Solution

$$\begin{aligned} f_x(x,y) &= e^{xy}\frac{\partial}{\partial x}(x) + x\frac{\partial}{\partial x}\left(e^{xy}\right) + \frac{\partial}{\partial x}\left(y^2\right) \\ &= e^{xy} + xye^{xy} + 0 \\ &= e^{xy}(1+xy) \end{aligned}$$

Now we evaluate $f_x(1,2)$, $f_x(1,2) = e^{1\times 2}(1+1\times 2) = 3e^2$

$$f_y(x,y) = x \times xe^{xy} + 2y = x^2e^{xy} + 2y$$

Now we evaluate $f_y(1,2)$, $f_y(1,2) = 1^2 \times e^{1\times 2} + 2\times 2 = e^2 + 4$

Example 4

Suppose that the production function $Q(x,y) = 2000x^{0.5}y^{0.5}$ is known. Determine the marginal productivity of labor and the marginal productivity of capital when 16 units of labor and 144 units of capital are used.

Solution

$$\frac{\partial Q}{\partial x} = 2000(0.5)x^{-0.5}y^{0.5} = \frac{1000y^{0.5}}{x^{0.5}}$$

$$\frac{\partial Q}{\partial y} = 2000(0.5)x^{0.5}y^{-0.5} = \frac{1000x^{0.5}}{y^{0.5}}$$

Substituting $x = 16$ and $y = 144$, we obtain

$$\left.\frac{\partial Q}{\partial x}\right|_{(16,144)} = \frac{1000(144)^{0.5}}{(16)^{0.5}} = \frac{1000\times 12}{4} = 3000 \text{ units}$$

and $$\left.\frac{\partial Q}{\partial y}\right|_{(16,144)} = \frac{1000(16)^{0.5}}{(144)^{0.5}} = \frac{1000\times 4}{12} = 333.33 \text{ units}$$

Thus we see that adding one unit of labor will increase production by about 3000 units and adding one unit of capital will increase production by about 333 units.

Example 5

It is estimated that the weekly output at a certain plant is given by the function $Q(x,y) = 1,200x + 500y + x^2y - x^3 - y^2$ units, where x is the number of skilled workers and y the number of unskilled workers employed at the plant. Currently the work force consists of 30 skilled workers, and 60 unskilled workers. Use maginal analysis to estimate the change in the weekly output that will reslt from the addition of 1 more skilled worker if the number of unskilled workers is not changed.

(Answer: $Q_x(30,60) = 2{,}100$ units.)

2.2 Second-Order Partial Derivatives

If $z = f(x,y)$, the partial derivative of f_x with respect to x is:

$$f_{xx} = (f_x)_x \text{ or } \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

The partial derivative of f_x with respect to y is

$$f_{xy} = (f_x)_y \text{ or } \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

The partial derivative of f_y with respect to x is

$$f_{yx} = (f_y)_x \text{ or } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

The partial derivative of f_y with respect to y is

$$f_{yy} = f_{yy} \text{ or } \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Example 6

Compute the four second-order partial derivatives of the function $f(x,y) = xy^3 + 5xy^2 + 2x + 1$.

Solution

Since $f_x = y^3 + 5y^2 + 2$, it follows that $f_{xx} = 0$ and $f_{xy} = 3y^2 + 10y$. Since $f_y = 3xy^2 + 10xy$, it follows that $f_{yx} = 3y^2 + 10y$ and $f_{yy} = 6xy + 10x$.

Example 7

Find all four second partial derivatives of $f(x,y) = \ln\left(x^2 + 4y\right)$, then find $f_{xx}(2,1/2)$.

Solution

We must find the first partial derivatives f_x and f_y before we can find the second partial derivatives.

$$f_x = \frac{1}{x^2 + 4y} \times 2x = \frac{2x}{x^2 + 4y}$$

$$f_y = \frac{1}{x^2 + 4y} \times 4 = \frac{4}{x^2 + 4y}$$

$$f_{xx} = \frac{\left(x^2 + 4y\right) \times 2 - 2x \times 2x}{\left(x^2 + 4y\right)^2} = \frac{-2x^2 + 8y}{\left(x^2 + 4y\right)^2}$$

$$f_{xy} = \frac{\left(x^2+4y\right)\times 0 - 2x\times 4}{\left(x^2+4y\right)^2} = \frac{-8x}{\left(x^2+4y\right)^2}$$

$$f_{yx} = \frac{\left(x^2+4y\right)\times 0 - 4\times 2x}{\left(x^2+4y\right)^2} = \frac{-8x}{\left(x^2+4y\right)^2}$$

$$f_{yy} = \frac{\left(x^2+4y\right)\times 0 - 4\times 4}{\left(x^2+4y\right)^2} = \frac{-16}{\left(x^2+4y\right)^2}$$

$$f_{xx}\left(2,\frac{1}{2}\right) = \frac{-2\times 2^2 + 8\left(\frac{1}{2}\right)}{\left(2^2+4\times\frac{1}{2}\right)^2} = \frac{-8+4}{\left(4+2\right)^2} = \frac{-4}{36} = -\frac{1}{9}$$

Example 8
Suppose the output Q at a factory depends on the amount K of capital invested in the plant and equipment and also on the size L of the labor force, measured in worker-hours. Give an economic interpretation of the sign of the second-order partial derivative $\frac{\partial^2 Q}{\partial L^2}$.

Solution
If $\frac{\partial^2 Q}{\partial L^2}$ is negative, the marginal product of labor $\frac{\partial Q}{\partial L}$ decreases as L increases. This implies that for a fixed level of capital investment, the effect on output of the addition of 1worker-hour of labor is greater when the work force is small than when the work force is large.
Similarly, if $\frac{\partial^2 Q}{\partial L^2}$ is positive, it follows that for a fixed level of capital investment, the effect on output of the addition of 1 worker-hour of labor is greater when the work force is larger than when it is small.

Remark The two partial derivatives f_{xy} and f_{yx} are sometimes called the **mixed second-order partial derivatives** of f and $f_{xy} = f_{yx}$.

3 The Chain Rule; Approximation by the Total Differential

3.1 Chain Rule for Partial Derivatives

Recall that if z is a differentiable function of x and x is a differentiable function of t, then z can be regarded as a differentiable function of t and the rate of change of z with respect to t is given by the chain rule

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt}$$

Here is the corresponding rule for functions of two variables.

Suppose z is a function of x and y, each of which is a function of t then z can be regarded as a function of t and

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

Remark 1

$$\frac{\partial z}{\partial x}\frac{dx}{dt}=\text{rate of change of } z \text{ with respect to } t \text{ for fixed } y.$$

$$\frac{\partial z}{\partial y}\frac{dy}{dt}=\text{rate of change of } z \text{ with respect to } t \text{ for fixed } x.$$

Example 1

Find $\frac{dz}{dt}$ if $z=x^2+3xy, x=2t+1,$ and $y=t^2$.

Solution

By the chain rule,

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=\left(2x+3y\right)\times 2+3x\times 2t$$

Which you can rewrite in terms of t by substituting $x=2t+1$ and $y=t^2$ to get

$$\frac{dz}{dt}=4(2t+1)+6t^2+3(2t+1)(2t)=18t^2+14t+4$$

Example 2

A health store carries two kinds of multiple vitamins, Brand A and Brand B. Sales figures indicate that if Brand A is sold for x dollars per bottle and Brand B for y dollars per bottle, the demand for Brand A will be

$$Q\left(x,y\right)=300-20x^2+30y \text{ bottles per month}$$

It is estimated that t months from now the price of Brand A will be

$$x=2+0.05t \text{ dollars per bottle}$$

and the price of Brand B will be

$$y=2+0.1\sqrt{t} \quad \text{dollars per bottle}$$

At what rate will the demand for Brand A be changing with respect to time 4 months from now?

Solution

Your goal is to find $\frac{dQ}{dt}$ when $t=4$. Using the chain rule, you get

$$\frac{dQ}{dt}=\frac{\partial Q}{\partial x}\frac{dx}{dt}+\frac{\partial Q}{\partial y}\frac{dy}{dt}$$

$$=-40x\left(0.05\right)+30\left(0.05t^{-1/2}\right)$$

when $t=4,\ x=2+0.05\times 4=2.2$

and hence,

$$\frac{dQ}{dt}=-40\times 2.2\times 0.05+30\times 0.05\times 0.5=-3.65$$

That is, 4 months from now the monthly demand for Brand A will be decreasing at the rate of 3.65 bottles per month.

3.2 The Total differential

Recall from chapter 2 that if y is a function of x,

$$\Delta y \simeq \frac{dy}{dx}\Delta x$$

where Δx is a small change in the variable x and Δy is the corresponding change in the function y. The expression $dy = \frac{dy}{dx}\Delta x$ that was used to approximate Δy was called the differential of y. Here is the analogous approximation formula for functions of two variables.

Approximation Formula

Suppose z is a function of x and y. If Δx denotes a small change in x and Δy a small change in y, the corresponding change in z is

$$\Delta z \simeq \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

Remark 2

$\frac{\partial z}{\partial x}\Delta x \simeq$ change in z due to the change in x for fixed y.

$\frac{\partial z}{\partial y}\Delta y \simeq$ change in z due to the change in y for fixed x.

The Total Differential

If z is a function of x and y, the total differential of z is

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

Example 3

At a certain factory, the daily output is $Q = 60K^{1/2}L^{1/3}$ units, where K denotes the capital investment measured in units of \$1,000 and L the size of the labor force measured in worker-hours. The current capital investment is \$ 900,000 and 1,000 and labor are used each day. Estimate the change in output that will result if capital investment is increased by \$1,000 and labor is increased by 2 worker-hours.

Solution

Apply the approximation formula with K=900, L=1000, $\Delta K = 1$, and $\Delta L = 2$ to get

$$\begin{aligned}\Delta Q &\simeq \frac{\partial Q}{\partial K}\Delta K + \frac{\partial Q}{\partial L}\Delta L \\ &= 30K^{-1/2}L^{1/3}\Delta K + 20K^{1/2}L^{-2/3}\Delta L \\ &= 30\times\left(\frac{1}{30}\right)\times 10\times 1 + 20\times 30\times\left(\frac{1}{100}\right)\times 2 \\ &= 22 \text{ units}\end{aligned}$$

That is, output will increase by approximately 22 units.

3.3 Approximation of Percentage Change

The **percentage change** of a quantity expresses the change in the quantity as a percentage of its size prior to the change. In perticular,

$$\text{Percentage change} = 100\frac{\text{change in quantity}}{\text{size of quantity}}$$

Approximation of Percentage Change

Suppose z is a function of x and y. If Δx denotes a small change in x and Δy a small change in y, the corresponding percentage change in z is

$$\text{Percentage change in } z = 100\frac{\Delta z}{z} \simeq 100\frac{\dfrac{\partial z}{\partial x}\Delta x + \dfrac{\partial z}{\partial y}\Delta y}{z}$$

Example 4

Use calculus to approximate the percentage by which the volume of a cylinder increases if the radius increases by 1 percent and the height increases by 2 percent.

Solution

The volume of a cylinder is given by the function $V(r,h) = \pi r^2 h,$ where r is the radius and h the height. The fact that r increases by 1 percent means that $\Delta r = 0.01r$ and the fact that h increases by 2 percent means that $\Delta h = 0.02h$. By the approximation formula for percentage change

$$\begin{aligned}
\text{Percentage change in } V &\simeq 100\frac{\dfrac{\partial V}{\partial r}\Delta r + \dfrac{\partial V}{\partial h}\Delta h}{V} \\
&= 100\frac{2\pi rh(0.01r) + \pi r^2(0.02h)}{\pi r^2 h} \\
&= 100\frac{0.02\pi r^2 h + 0.02\pi r^2 h}{\pi r^2 h} \\
&= 100\frac{0.04\pi r^2 h}{\pi r^2 h} = 4 \text{ percent}
\end{aligned}$$

Example 5

At a certain factory, output is given by the Cobb-Douglas production function $Q(K,L) = AK^{\alpha}L^{1-\alpha}$, where A and α are positive constants with $0 < \alpha < 1$, and where K denotes the capital investment and L the size of the labor force. Use calculus to estiamate the percentage by which output will change if both capital and labor are increased by 1 percent. (Answer: 1%)

4 Relative Maxima and Minima

In geometric terms, a **relative maximimum** of a function $f(x,y)$ is a peak, a point on the surface $z = f(x,y)$ that is higher than any nearby point on the surface. A **relative minimum** is the bottom of a valley, a point that is lower than any nearby point on the surface.

Critical Points and Relative Extrema

A point (a,b) in the domain of $f(x,y)$ for which both $f_x(a,b)=0$ and $f_y(a,b)=0$ is said to be a **critical point** of *f*.

If the first-order partial derivatives of *f* are defined at all points in some region in the *xy*-plane, then the relative extrema of *f* in the region can occur only at critical points.

However, not every critical point is a relative maximum or a relative minimum. A critical point that is neither a relative maximum nor a relative minimum is called a **saddle point.** The below procedure involving second-order partial derivatives is used to decide whether a given critical point is a relative maximum, a relative minimum, or a saddle point.

The second Partials Test

Suppose that $f(a,b)$ is a critical point of the function $f(x,y)$. Let

$$D = f_{xx}(a,b)\,f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

If $D<0$, then *f* has a saddle point at (a,b)

If $D>0$, and $f_{xx}(a,b)<0$, then *f* has a relative maximum at (a,b).

If $D>0$, and $f_{xx}(a,b)>0$, then *f* has a relative minimum at (a,b).

If $D=0$, then test is inconclusive and *f* may have either a relative extremum or a saddle point at (a,b).

Example 1

Classify the critical point of the function $f(x,y)=x^2+y^2$.

(Answer: The critical point $(0,0)$ is the relative minimum.)

Example 2

Classify the critical point of the function $f(x,y)=y^2-x^2$.

(Answer: The saddle point $(0,0)$)

Example 3

Find all the local minima, local maxima, and saddle points for the function

$$f(x,y)=x^2-xy+y^2-9x+5$$

Solution

Find the first partial derivatives f_x and f_y.

$$f_x(x,y)=2x-y-9 \quad \text{and} \quad f_y(x,y)=-x+2y$$

Now, to find any critical point, solve the following system.

$$\begin{cases} 2x-y-9=0 \\ -x+2y=0 \end{cases} \Rightarrow x=6, y=3$$

The only critical point is $(6,3)$.

The second partials are

$$f_{xx}(x,y)=2,\ f_{xy}(x,y)=-1 \quad \text{and} \quad f_{yy}(x,y)=2$$

In this case each of the second partials is a constant and will have that constant value at $(6,3)$. Thus,

$$D = f_{xx}(6,3)\, f_{yy}(6,3) - \left[f_{xy}(6,3) \right]^2$$
$$= 2\times 2 - (-1)^2 = 4 - 1 = 3 > 0$$

Since $D > 0$, we check the sign of $f_{xx}(6,3)$ to determine whether $(6,3)$ yields a local minimum or a local maximum. And since $f_{xx}(6,2) = 2 > 0$, the critical point $(6,3)$ yields a local minimum value. This value is $f(6,3) = 6^2 - 6\times 3 + 3^2 - 9\times 6 + 5 = -22$

Example 4

A company produces and sells two styles of umbrellas. One style sells for \$20 each and the other sells for \$25 each. The company has determined that if x thousand of the first style and y thousand of the second style are produced, then the total cost in thousands of dollars is given by the function

$$C(x,y) = 3x^2 - 3xy + \frac{3}{2}y^2 + 32x - 29y + 70$$

How many of each style of umbrella should the company produce and sell in order to maximize profit?

Solution

Since x thousand umbrellas sell for \$20 each and y thousand umbrellas sell for \$25 each, the revenue function (in thousands of dollars) is given by $R(x,y) = 20x + 25y$

Thus the profit function is

$$P(x,y) = R(x,y) - C(x,y)$$
$$= 20x + 25y - \left(3x^2 - 3xy + \tfrac{3}{2}y^2 + 32x - 29y + 70\right)$$
$$= -3x^2 + 3xy - \tfrac{3}{2}y^2 - 12x + 54y - 70$$

The first partial derivatives are

$$P_x = -6x + 3y - 12 \text{ and } P_y = 3x - 3y + 54$$

Now solve the following system of equations.

$$\begin{cases} -6x + 3y - 12 = 0 \\ 3x - 3y + 54 = 0 \end{cases} \Rightarrow x = 14, y = 32$$

The company will make the maximum profit if it produces and sells 14,000 of the first style of umbrella and 32,000 of the second style. (The student can verify (with the D-Test) that the profit is indeed a maximum at (14, 32).)

Example 5

The only grocery store in a small rural community carries two brands of frozen orange juice, a local brand that it obtains at the cost of 30 cents per can and a well-known national brand that it obtains at the cost of 40 cents per can. The grocer estimates that if the local brand is sold for x cents per can and the national brand for y cents per can, approximately $70 - 5x + 4y$ cans of the local brand and $80 + 6x - 7y$ cans of the national brand will be sold each day. How should the grocer price each brand to maximize the profit from the sale of the juice? (Assume that the absolute maximum and the relative maximum of the profit function are the same.)

(Answer: $x = 53, y = 55$)

5 Lagrange Multipliers

5.1 Contrained Optimization Problems

In many applied problems, a function of two variables is to be optimized subject to a restriction or **constraint** on the varaibles. For, example, an editor constrained to stay within a fixed budget of $60,000, may wish to decide how to divide this money between development and promotion in order to maximize the future sales of a new book. If x denotes the amount of money allocated to development, y the amount allocated to promotion, and $f(x, y)$ the corresponding number of books that will be sold, the editor would like to maximize the sales function $f(x, y)$ subject to the budgetary constraint that $x + y = 60,000$. To deal with this problem, we use a technique called the **method of Lagrange multipliers.**

The Method of Lagrange Multipliers

Suppose $f(x, y)$ and $g(x, y)$ are functions whose first-order-partial derivatives exist. To find the relative maximum and relative minimum of $f(x, y)$ subject to the constraint that $g(x, y) = K$ for some constant K, introduce a new variable λ (the Greek letter lambda) and solve the following three equations simultaneously:

$$\begin{cases} f_x(x, y) = \lambda g_x(x, y) \\ f_y(x, y) = \lambda g_y(x, y) \\ g(x, y) = K \end{cases}$$

The desired relative extrema will be found among the resulting points (x, y).

Example 1

Find the maximum and minimum values of the function $f(x, y) = xy$ subject to the constraint $x^2 + y^2 = 8$.

Solution

Let $g(x, y) = x^2 + y^2$ and use the partial derivatives

$$f_x = y,\ f_y = x,\ g_x = 2x, \text{ and } g_y = 2y$$

to get the three Lagragnge equations

$$y = 2\lambda x,\ \ x = 2\lambda y,\ \text{ and } x^2 + y^2 = 8$$

The first two equations can be rewritten as

$$2\lambda = \frac{y}{x} \text{ and } 2\lambda = \frac{x}{y}$$

which implies that

$$\frac{y}{x} = \frac{x}{y} \text{ or } x^2 = y^2$$

Now substitute $x^2 = y^2$ into the third equation to get

$$2x^2 = 8 \text{ or } x = \pm 2$$

If $x = 2$, it follows from the equation $x^2 = y^2$ that $y = 2$ or $y = -2$. Similarly, if $x = -2$, it follows that $y = 2$ or $y = -2$. Hence, the four points at which the constrained extrema can occur are $(2,2),(2,-2),(-2,2)$, and $(-2,-2)$. Since

$$f(2,2) = f(-2,-2) = 4 \text{ and } f(2,-2) = f(-2,2) = -4$$

it follows that when $x^2 + y^2 = 8$, the maximum value of $f(x,y)$ is 4, which occurs at the points $(2,2)$ and $(-2,-2)$ and the minimum value is -4 which occurs at $(2,-2)$ and $(-2,2)$.

Example2

An editor has been allocated \$60,000 to spend on the development and promotion of a new book. It is estimated that if x thousand dollars is spent on development and y thousand on promotion, approximately $f(x,y) = 20x^{3/2}y$ copies of the book will be sold. How much money should the editor allocate to development and how much to promotion in order to maximize sales?

Solution:

The goal is to maximize the function $f(x,y) = 20x^{3/2}y$ subject to the constraint $g(x,y) = 60$, where $g(x,y) = x + y$. The corresponding Lagrange equations are

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = K \end{cases} \Rightarrow \begin{cases} 30x^{1/2}y = \lambda, & (1) \\ 20x^{3/2} = \lambda, & (2) \\ x + y = 60, & (3) \end{cases}$$

From (1) and (2) you get

$$30x^{1/2}y = 20x^{3/2}$$

$$y = \frac{2}{3}x$$

Substituting this expression into the (3) you get

$$x + \frac{3}{2}x = 60 \text{ or } \frac{5}{2}x = 60$$

From which it follows that

$$x = 36 \text{ and } y = 24$$

That is, to maximize sales, the editor should spend \$36,000 on development and \$ 24,000 on promotion. If this is done, approximately $f(36,24) = 103,680$ copies of the book will be sold.

Example 3

A consummer has \$600 to spend on two commodities, the first of which costs \$20 per unit and the second \$30 per unit. Suppose that the utility derived by the consumer from x units of the first commodity and y units of the second commodity is given by the **Cobb-Douglas utility function** $U(x,y) = 10x^{0.6}y^{0.4}$. How many units of each commodity should the consumer buy to maximize utility? (A **utility function**

$U(x,y)$ measures the total satisfaction or **utility** the consumer receives from having x units of the first commodity and y units of the second.) (Answer: $x=18, y=8$)

5.2 The Lagrange Multiplier

In some problems, we need to compute the **Lagrange multiplier** λ since it has the following useful interpretation.

Suppose M is the maximum (or minimum) value of $f(x,y)$ subject to the constraint $g(x,y)=K$. The Lagrange multiplier λ is the rate of change of M with respect to K. That is,

$$\lambda = \frac{dM}{dK}$$

Hence,

$\lambda \simeq$ Change in M resulting from a 1-unit increase in K.

Example 4

Suppose the editor in Example 2 is allotted \$60,200 instead of \$ 60,000 to spend development and promotion of the new book. Estimate how the additional \$ 200 will affect the maximum sales level.

Solution

In Example 2, you solved the three Lagrange equations

$$30x^{1/2}y = \lambda$$
$$20x^{3/2} = \lambda$$
$$x + y = 60$$

to conclude that the maximum value M of $f(x,y)$ subject to the constrain $x+y=60$ occurred when $x=36$ and $y=24$. To find λ substitute these values of x and y into the first or second Lagrange equation. Using the second equation, you get

$$\lambda = 20(36)^{3/2} = 4{,}320$$

The goal is to estimate the change ΔM in the maximal sales that will result from an increase of $\Delta K = 0.2$ (thousand dollars) in the available funds.

Since $\lambda = \dfrac{dM}{dK}$, the one-variable approximation formula, gives

$$\Delta M \approx \frac{dM}{dK}\Delta K = \lambda \Delta K = 4{,}320(0.2) = 864$$

That is, maximal sales of the book will increase by approximately 864 copies if the budget is increased from \$ 60,000 to \$60,200 and the money is allocated optimally.

Example 5

Suppose the consumer in example 3 has \$601 instead of \$600 to spend on the two commodities. Estimate how the additional \$1 will affect the maximum utility.

(Answer: $\Delta M \simeq 0.22$)

Exercises

1 Using x skilled workers and y unskilled workers, a manufacturer can produce $Q(x,y)=10x^2y$ units per day. Currently there are 20 skilled workers and 40 unskilled workers on the job.

a. How many units are currently being produced each day?
b. By how much will the daily production level change if 1 more skilled worker is added to the current work force?
c. By how much will the daily production level change if 1 more unskilled worker is added to the current work force?
d. By how much will the daily production level change if 1 more skilled worker *and* 1 more unskilled worker are added to the current work force?

ដោយប្រើកម្មករជំនាញចំនួន x និងកម្មករមិនជំនាញចំនួន y ផលិតករម្នាក់អាចផលិតបានចំនួន $Q(x,y)=10x^2y$ ឯកតា ក្នុងមួយថ្ងៃ ។ ឥឡូវនេះមានកម្មករជំនាញចំនួន20នាក់ និងកម្មករមិនជំនាញចំនួន40កំពុងធ្វើការងារ ។ **ក.** តើក្នុងមួយថ្ងៃៗ គេអាចផលិតបានប៉ុន្មានឯកតា? **ខ.** តើបរិមាណផលិតប្រចាំថ្ងៃប្រែប្រួលក្នុងទំហំប៉ុន្មាន បើគេបន្ថែមកំលាំងកម្មករជំនាញចំនួន 1 នាក់ទៀតទៅលើចំនួនដែលមាន ស្រាប់? **គ.** តើបរិមាណផលិតប្រចាំថ្ងៃប្រែប្រួលក្នុងទំហំប៉ុន្មាន បើគេបន្ថែមកំលាំងកម្មករមិនជំនាញចំនួន 1 នាក់ទៀតទៅលើចំនួនដែលមានស្រាប់? **ឃ.** តើបរិមាណផលិតប្រចាំថ្ងៃប្រែប្រួលក្នុងទំហំប៉ុន្មាន បើគេបន្ថែមកំលាំងកម្មករជំនាញចំនួន1នាក់ និងកំលាំងកម្មករមិនជំនាញ 1 មួយនាក់ទៀតដែរទៅលើចំនួនដែលមានស្រាប់?

(Answer: **a**. 160,000 units per day, **b**. 16,400 units per day, **c**. 4000 units per day **d**. 20,810 units per day)

2 A manufacturer can produce electric typewriters at a cost of \$80 apiece and manual typewriters at a cost of \$20 apiece.

a. Express the manufacturer's total monthly production cost as a function of the number of electric typewriters and the number of manual typewriters produced.
b. Compute the total monthly cost if 500 electric and 800 manual typewriters are produced.
c. The manufacturer wants to increase the output of electric typewriters by 50 a month from the level in part b. What corresponding change should be made in the monthly output of manual typewriters so that the total monthly cost will not change?

ផលិតករម្នាក់អាចផលិតអង្គុលីលេខអគ្គិសនីដោយប្រើថ្លៃដើម 80ដុល្លាក្នុងមួយគ្រឿង និងអង្គុលីលេខដៃដោយប្រើថ្លៃដើម 20ដុល្លាក្នុងមួយគ្រឿង ។ **ក.** ចូរសរសេរអនុគមន៍ថ្លៃដើមផលិតកម្មប្រចាំខែរបស់ផលិតករជាអនុគមន៍ទៅនឹងចំនួនអង្គុលីលេខទាំងពីរប្រភេទ ដែលបានផលិត ។ **ខ.** គណនាថ្លៃដើមសរុបប្រចាំខែ បើសិនជាអង្គុលីលេខអគ្គិសនីចំនួន 500 និងអង្គុលីលេខដៃចំនួន 800គ្រឿងត្រូវផលិត ។ **គ.** ផលិតករចង់បង្កើនផលិតផលអង្គុលីលេខអគ្គិសនី ចំនួន50គ្រឿងបន្ថែមទៀតក្នុងមួយខែពីលើកម្រិតដែលបានបញ្ជាក់នៅក្នុងសំណួរ **ខ.** ។ តើចំនួនផលិតផលអង្គុលីលេខដៃដែលត្រូវផលិតប្រចាំខែត្រូវប្រែប្រួលដូចម្តេច ដើម្បីរក្សាកុំឱ្យ

ការប្រែប្រួលចំណាយលើថ្លៃដើមសរុបប្រចាំខែ?

3 A paint store carries two brands of latex paint. Sales figures indicate that if the first brand is sold for x_1 dollars per gallon and the second for x_2 dollars per gallon, the demand for the first brand will be $D_1(x_1, x_2) = 200 - 10x_1 + 20x_2$ gallons per month and the demand for the second brand will be $D_2(x_1, x_2) = 100 + 5x_1 - 10x_2$ gallons per month.

a. Express the paint store's total monthly revenue from the sale of the paint as a function of the prices x_1 and x_2.

b. Compute the revenue in part a. if the first brand is sold for \$6 per gallon and the second for \$5 per gallon.

ហាងលក់គ្រឿងសំណង់មួយមានលក់ថ្នាំលាបចំនួនពីរប្រភេទ។ តាមតួលេខនៃការលក់បានបង្ហាញថាបើថ្នាំលាបប្រភេទទី១លក់ថ្លៃ x_1 ក្នុងមួយហ្គាឡុង និងថ្នាំលាបប្រភេទទី២លក់ថ្លៃ x_2 ដុល្លាក្នុងមួយហ្គាឡុង នោះតម្រូវការថ្នាំប្រភេទទី១ស្មើទៅនឹង $D_1(x_1, x_2) = 200 - 10x_1 + 20x_2$ ក្នុងមួយខែ និងតម្រូវការថ្នាំប្រភេទទី២ស្មើទៅនឹង $D_2(x_1, x_2) = 100 + 5x_1 - 10x_2$ ក្នុងមួយខែ។ **ក.** ចូរសរសេរកន្សោមតាមឱ្យប្រាក់ចំណូលសរុបប្រចាំខែរបស់ហាងនោះជាអនុគមន៏ទៅនឹងថ្លៃ x_1 និង x_2 ។ **ខ.** ដោយប្រើអនុគមន៏ប្រាក់ចំណូលនេះ ចូរគណនាប្រាក់ចំណូល បើថ្នាំប្រភេទ១ លក់ថ្លៃ 6ដុល្លាក្នុងមួយហ្គាឡុង និងថ្នាំប្រភេទ២លក់ថ្លៃ 5ដុល្លាក្នុង១ហ្គាឡុង។

4 The output at a certain factory is $Q(K, L) = 120K^{2/3}L^{1/3}$ units, where K is the capital investment measured in units of \$1,000 and L the size of the labor force measured in worker-hours.

a. Compute the output if the capital investment is \$125,000 and the size of the labor force is 1,331 worker-hours.

b. What will happen to the output in part a. if both the level of capital investment and the size of the labor force are cut in half?

ចំនួនផលិតផលចេញពីរោងចក្រមួយកំណត់ដោយអនុគមន៏ $Q(K, L) = 120K^{2/3}L^{1/3}$ ឯកតា ដែលKគឺជាវិនិយោគមូលធន ដែលខ្នាតគិតជាពាន់ដុល្លា ហើយLជាទំហំកំលាំងពលកម្ម ខ្នាតគិតជា**ម៉ោងកម្មករ**[9] ។ **ក.** គណនាបរិមាណផលិតផលដែលផលិតបានបើគេមានវិនិយោគមូលធន 125000ដុល្លា និងទំហំកំលាំងពលកម្ម 1331ម៉ោងកម្មករ។ **ខ.** តើបរិមាណផលិតផលក្នុងសំណួរ**ក.** នឹងប្រែប្រួលដូចម្តេច បើទាំងទំហំកំលាំងពលកម្ម ទាំងវិនិយោគមូលធនត្រូវកាត់បន្ថយនៅសល់ពាក់កណ្តាល?

5 At a certain factory, the daily output is $Q(K, L) = 60K^{1/2}L^{1/3}$ units, where K denotes the capital investment measured in unit of \$1,000 and L size of labor force measured in worker-hours. Suppose that the current capital investment is \$900,000 and that 1,000 worker-hours of labor are used each day. Use marginal

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[9] ជាខ្នាតសម្រាប់វាស់នូវបរិមាណដែលសម្រេចបានដោយកម្មករម្នាក់ក្នុងរយៈពេលមួយម៉ោង។

analysis to estimate the effect of an additional campital investment of \$1,000 on the daily output if the size of the labor force is not changed. (Answer: 10 units)

នៅក្នុងរោងចក្រមួយ ចំនួនផលិតផលប្រចាំថ្ងៃកំណត់ដោយអនុគមន៍ $Q(K,L)=60K^{1/2}L^{1/3}$ ដែលKគឺជាវិនិយោគមូលធន ដែលខ្នាតគិតជាពាន់ដុល្លា ហើយLជាទំហំកំលាំងពលកម្ម ខ្នាតគិតជា**ម៉ោងកម្មករ** ។ ឧបមាថាវិនិយោគមូលធនបច្ចុប្បន្នមាន 900000ដុល្លា ហើយទំហំកំលាំងពលកម្មមាន 1000ម៉ោងកម្មករ ត្រូវបានប្រើប្រាស់រាល់ៗថ្ងៃ ។ ប្រើការវិភាគម៉ាជីន ដើម្បីប៉ាន់ស្មានអំពីឥទ្ធិពលនៃវិនិយោគមូលធនចំនួន1000 ដុល្លាបន្ថែម មានទៅលើផលិតកម្មប្រចាំថ្ងៃ បើទំហំកំលាំងពលកម្ម មិនប្រែប្រួល ។

6 A grocer's daily profit from the sale of two brands of orange juice is

$$P(x,y)=(x-30)(70-5x+4y)+(y-40)(80+6x-7y)$$

cents, where *x* is the price per can of the first brand and *y* is the price per can of the second. Currently the first brand sells for 50 cents per can and the second for 52 cents per can. Use marginal analysis to estimate the change in the daily profit that will result if the grocer raises the price of the second brand by 1 cent per can but keeps the price of the first brand unchanged. (Answer: \$0.12)

ប្រាក់ចំណេញប្រចាំថ្ងៃរបស់អាជីវករម្នាក់ពីការលក់ទឹកក្រូចពីរប្រភេទ កំណត់ដោយអនុគមន៍ $P(x,y)=(x-30)(70-5x+4y)+(y-40)(80+6x-7y)$ សេន ។ x គឺជាថ្លៃលក់ក្នុងមួយកំប៉ុងសម្រាប់ទឹកក្រូចប្រភេទទី១ និង yគឺជាថ្លៃលក់ក្នុងមួយកំប៉ុងសម្រាប់ទឹកក្រូចប្រភេទទី២ ។ បច្ចុប្បន្ននេះទឹកក្រូចប្រភេទទី១ លក់ថ្លៃ 50សេនក្នុងមួយកំប៉ុង និងទឹកក្រូចប្រភេទទី២ លក់ថ្លៃ52 សេនក្នុងមួយកំប៉ុង ។ ប្រើការវិភាគម៉ាជីន ដើម្បីប៉ាន់ស្មាននូវបរិមាណប្រែប្រួលនៃប្រាក់ចំណេញប្រចាំថ្ងៃដែលកើតឡើង បើសិនជាអាជីវករបង្កើនថ្លៃទឹកក្រូចប្រភេទទី២ ចំនួន 1 សេនក្នុងមួយកំប៉ុងប៉ុន្តែរក្សាថ្លៃទឹកក្រូចប្រភេទទី1 នៅដដែល ។

7 Compute all the second-order partial derivatives of the given function

គណនាដេរីវេដោយផ្នែកលំដាប់២ទាំងអស់របស់អនុគមន៍ខាងក្រោម

a. $f(x,y)=5x^4y^3+2xy$

b. $f(x,y)=\dfrac{x+1}{y-1}$

c. $f(x,y)=e^{x^2y}$

d. $f(x,y)=\sqrt{x^2+y^2}$

e. $f(x,y)=x^2ye^x$

8 Use chain rule to find $\dfrac{dz}{dt}$. Check your answer by writing *z* as a function of *t* and differentiating directly with respect to *t*. ប្រើវិធានបណ្ដាក់ដើម្បីរក $\dfrac{dz}{dt}$ ។ ផ្ទៀងផ្ទាត់ចម្លើយតាមរយៈការសរសេរzជាអនុគមន៍នៃ t រួចរកដេរីវេធៀបនឹង t ដោយផ្ទាល់ ។

a. $z=x+2y; x=3t, y=2t+1$

b. $z=3x^2+xy; x=t+1, y=1-2t$

c. $z=(2x+3y)^2; x=2t, y=3t$

9 Use chain rule to find $\dfrac{dz}{dt}$ for the specified value of t

ប្រើវិធានបណ្តាក់ដើម្បីរក $\frac{dz}{dt}$ ទៅតាមតម្លៃtដែលគេបានបញ្ជាក់ ។

a. $z = 2x + 3y; x = t^2, y = 5t; t = 2$ **b.** $z = x^2 y; x = 3t + 1, y = t^2 - 1; t = 1$

c. $z = x^{1/2} y^{1/3}; x = 2t, y = 2t^2; t = 2$

11 Using x skilled workers and y unskilled workers, a manufacturer can produce $f(x, y) = 10xy^{1/2}$ units. Currently the manufacturer uses 30 hours of skilled labor and 36 hours of unskilled labor and is planning to use 1 additional hour of skilled labor. Use calculus to estimate the corresponding change that the manufacturer should make in the level of unskilled labor so that the total output will remain the same. (Answer: −2.4)

ដោយប្រើកម្មករជំនាញចំនួន xម៉ោង និងកម្មករមិនជំនាញចំនួន yម៉ោង ផលិតករម្នាក់អាចផលិតបាន $f(x, y) = 10xy^{1/2}$ ឯកតា ។ បច្ចុប្បន្ននេះ ផលិតករប្រើកម្មករជំនាញ30ម៉ោង និងកម្មករមិនជំនាញ36 ម៉ោង ។ ផលិតករគ្រោងនឹងប្រើប្រាស់កំលាំងពលកម្មជំនាញ១ម៉ោងបន្ថែម ។ ចូរប៉ាន់ស្មានបរិមាណប្រែប្រួលដែលត្រូវធ្វើឡើងចំពោះកំលាំពលកម្មមិនជំនាញ យ៉ាងណាដើម្បីឱ្យផលិតផលសរុបនៅរក្សាដដែល ។

12 At a certain factory, output Q is related to inputs x and y by the function $Q = 2x^3 + 3x^2 y + y^3$. If the current levels of input are x = 20 and y = 10, use calculus to estimate the change in input x that should be made to offset an increase of 0.5 unit in input y so that output will be maintained at it current level.

(answer:−0.21)

នៅក្នុងរោងចក្រមួយ output Qជាប់ទាក់ទងទៅនឹង input x និងyតាមរយៈអនុគមន៍ $Q = 2x^3 + 3x^2 y + y^3$ ។ បរិមាណបច្ចុប្បន្នរបស់ input xគឺ x = 20និង input yគឺ y = 10 ។ ប្រើរូបមន្តតម្លៃប្រហែល ដើម្បីប៉ាន់ស្មានអំពីបរិមាណប្រែប្រួលរបស់ input x ដើម្បីទូទាត់នឹងបរិមាណកំណើនរបស់ input y ក្នុងកំរិ 0.5ឯកតា យ៉ាងណាដើម្បីឱ្យបរិមាណ outputស្ថិតនៅក្នុងកំរិតដដែល ។

13 Suppose the utility derived by a consumer from x units of one comodity and y units of a seccond commodity is given by the utility function $U(x, y) = (x+1)(y+2)$. The consumer currently owns $x = 25$ units of the first commodity and $y = 8$ units of the second. Use calculus to estimate how many units of the first commodity the consumer could substitute for 1 unit of the second commodity without affecting total utility. (about 2.6).

ឧបមាថាការប្រើប្រាស់អតិថិជនម្នាក់ចេញពី**របស់ប្រើប្រាស់ទី១**ចំនួនxឯកតា និង**របស់ប្រើប្រាស់ទី២**ចំនួន yឯកតាត្រូវ កំណត់ដោយអនុគមន៍ការប្រើប្រាស់ $U(x, y) = (x+1)(y+2)$ ។ អតិថិជនបច្ចុប្បន្នមានរបស់ប្រើប្រាស់ទី១ ចំនួន $x = 25$ឯកតា និងរបស់ប្រើប្រាស់ទី២ចំនួន $y = 8$ឯកតា ។ ប្រើរូបមន្តតម្លៃប្រហែល ចូរប៉ាន់ស្មានថាតើបរិមាណ**របស់ប្រើប្រាស់ទី១**ចំនួនប៉ុន្មានឯកតាដែលអាចជំនួសបរិមាណ**របស់ប្រើប្រាស់ទី២** ចំនួន 1ឯកតា ដើម្បីកុំឱ្យមានការប្រែប្រួលដល់បរិមាណការប្រើប្រាស់សរុប?

15 The output at a certain plant is $Q(x, y) = 0.08x^2 + 0.12xy + 0.03y^2$ units per day, where x is the number of hours of skilled labor used and y is the number of hours of unskilled labor used. Currently 80 hours of skilled labor and 200 hours of unskilled labor are used each day. Used the total differential of Q to estimate the

change in output that will result if and additional $\frac{1}{2}$hour of skilled labor is used along with an additional 2 hours of unskilled labor. (Answer: 61.6)

រោងចក្រមួយផលិតបាន $Q(x,y)=0.08x^2+0.12xy+0.03y^2$ ឯកតា ក្នុងមួយថ្ងៃៗ ដែលxគឺជាចំនួនម៉ោងដែលប្រើប្រាស់កំលាំងពលកម្មជំនាញ និងyគឺជាចំនួនម៉ោងដែលប្រើប្រាស់កំលាំងពលកម្មមិនជំនាញ ។ បច្ចុប្បន្នគេប្រើកំលាំងពលកម្មជំនាញចំនួន 80ម៉ោង និងកំលាំងពលកម្មមិនជំនាញចំនួន 200ម៉ោងជារៀងរាល់ថ្ងៃ ។ ប្រើរូបមន្តឌីផេរ៉ង់ស្យែលសរុបរបស់អនុគមន៍ Qដើម្បីប៉ាន់ស្មានពីបរិមាណប្រែប្រួលនៃផលិតផល ដែលកើតពីការប្រើប្រាស់បន្ថែមនូវកំលាំងពលកម្មជំនាញចំនួនកន្លះម៉ោងនិងកំលាំងពលកម្មមិនជំនាញចំនួន2ម៉ោង ។

16 At a certain factory, the output is $Q=120K^{1/2}L^{1/3}$ units, where K denotes the capital investment measured in units of \$1,000 and L the size of the labor force measured in worker-hours. The current capital investment is \$400,000 and 1,000 worker-hours of labor are currently used. Use the total differential of Q to estimate the change in output that will result if capital investment is increased by \$500 and labor is increased by 4 worker-hours.

ផលិតផលចេញពីរោងចក្រមួយកំណត់ដោយអនុគមន៍$Q=120K^{1/2}L^{1/3}$ ឯកតា ដែលKតាងឱ្យវិនិយោគមូលធន គិតជាពាន់ដុល្លា ហើយLគឺជាទំហំកំលាំងពលកម្មដែលគិតជាម៉ោងកម្មករ ។

បច្ចុប្បន្នគេប្រើវិនិយោគមូលធនចំនួន 400000ដុល្លា និងកំលាំងពលកម្មចំនួន1000ម៉ោងកម្មករ ។ ចូរប្រើឌីផេរ៉ង់ស្យែលសរុបរបស់អនុគមន៍Qចូរប៉ាន់ស្មានបរិមាណប្រែប្រួលរបស់ផលិតផលដែលកើតឡើងដោយសារគេបង្កើនវិនិយោគមូលធនចំនួន 500ដុល្លា និងបង្កើនកំលាំងពលកម្មចំនួន 4 ម៉ោងកម្មករ ។

17 A grocer's daily profit from the sale of two brands of orange juice is $P(x,y)=(x-30)(70-5x+4y)+(y-40)(80+6x-7y)$cents, where x is the price per can of the first brand and y is the price per can of the second. Currently the first brand sells for 50 cents per can and the second for 52 cents per can. Use the total differential of P to estimate the change in the daily profit that will result if the grocer raises the price of the first brand by 1 cent per can and raises the price of the second brand by 2 cents per can. (Answer: $\simeq$ 24 cents)

ប្រាក់ចំណេញប្រចាំថ្ងៃរបស់អ្នកលក់គ្រឿងទេសម្នាក់ពីការលក់ទឹកក្រូចពីរប្រភេទកំណត់ដោយអនុគមន៍ $P(x,y)=(x-30)(70-5x+4y)+(y-40)(80+6x-7y)$សេន ដែល$x$ជាថ្លៃក្នុងមួយកំប៉ុងនៃទឹកក្រូចប្រភេទទី១ ហើយyគឺជាថ្លៃក្នុងមួយកំប៉ុងនៃទឹកក្រូចប្រភេទទី២ ។ បច្ចុបន្ននេះទឹកក្រូចប្រភេទទី១លក់ថ្លៃ 50សេនក្នុងមួយកំប៉ុង និងទឹកក្រូចប្រភេទទី២លក់ថ្លៃ52សេនក្នុងមួយកំប៉ុង ។ ប្រើឌីផេរ៉ង់ស្យែលសរុបរបស់អនុគន៏Pចូរប៉ាន់ស្មានបរិមាណប្រែប្រួលនូវប្រាក់ចំណេញប្រចាំថ្ងៃដែលកើតឡើងដោយសារអ្នកលក់បង្កើនថ្លៃទឹកក្រូចទីប្រភេទទី១ ចំនួន1 សេនក្នុងមួយកំប៉ុង និងបង្កើនថ្លៃទឹកក្រូចប្រភេទទី២ចំនួន២សេនក្នុងមួយកំប៉ុង ។

18 An editor estimates that if x thousand dollars is spent on development and y thousand on promotion, approximately $Q(x,y)=20x^{3/2}y$ copies of a new book will be sold. Current plans call for the expenditure of \$36,000 on development and \$25,000 on promotion. Use the total differential of Q to estimate the change in sales that will result if the amount spent on development is increased by \$500 and the amount spent on promotion is decreased by \$500.

ពណ្ណាធិការម្នាក់ប៉ាន់ស្មានថាបើសិនជាគាត់ចំណាយ xពាន់ដុល្លាទៅលើការបោះពុម្ព និងyពាន់ដុល្លាទៅលើការផ្សព្វផ្សាយសៀវភៅថ្មី នោះសៀវភៅប្រហែល $Q(x,y) = 20x^{3/2}y$ច្បាប់នឹងត្រូវលក់ដាច់អស់ ។ នៅក្នុងគំរោងបច្ចុប្បន្ននេះគាត់ចំណាយ 36000ដុល្លាទៅលើការបោះពុម្ព និង25000ដុល្លាទៅលើការផ្សព្វផ្សាយ ។ ចូរប្រើឌីផេរ៉ង់ស្យែលសរុបរបស់អនុគមន៍ Qដើម្បីប៉ាន់ស្មានបរិមាណប្រែប្រួលនៃការលក់ដែលកើតឡើងដោយសារការបង្កើនចំណាយលើការបោះពុម្ពក្នុងបរិមាណ 500ដុល្លា និងការបន្ថយការចំណាយលើការផ្សព្វផ្សាយក្នុងបរិមាណ 500ដុល្លា ។

19 At a certain factory, the daily output is $Q(K,L) = 60K^{1/2}L^{1/3}$ units, where K denotes the capital investment and L the size of the labor force. Use calculus to estimate the percentage by which the daily output will change if capital investment is increased by 1 percent and labor by 2 percent.

នៅឯរោងចក្រមួយ output ប្រចាំថ្ងៃកំណត់ដោយអនុគមន៍ $Q(K,L) = 60K^{1/2}L^{1/3}$ ឯកតា ដែលKគឺជាវិនិយោគមូលធន ហើយLគឺជាទំហំកំលាំងពលកម្ម ។ ដោយប្រើរូបមន្តតម្លៃប្រហែល ចូររកតម្លៃប្រហែលនៃបរិមាណប្រែប្រួលជាភាគរយ បើសិនជាវិនិយោគមូលធនត្រូវបានបង្កើន 1ភាគរយ និងកំលាំងពលកម្ម 2ភាគរយ ។

20 Find the critical points of the given functions and classify them as relative maxima, relative minima, or saddle points.

ចំពោះអនុគមន៍ខាងក្រោម ចូររកចំណុចទិតៀន រួចបញ្ជាក់ប្រាប់អំពីប្រភេទរបស់ចំណុចទិតៀនទាំងនោះថាតើវាជាចំណុចអតិបរមាធៀប អប្បបរមាធៀប ឬជាចំណុចកែបសេះ ។

a. $f(x,y) = 5 - x^2 - y^2$
b. $f(x,y) = 2x^2 - 3y^2$
c. $f(x,y) = xy$
d. $f(x,y) = xy + \frac{8}{x} + \frac{8}{y}$
e. $f(x,y) = 2x^3 + y^3 + 3x^2 - 3y - 12x - 4$
f. $f(x,y) = -x^4 - 32x + y^3 - 12y + 7$

21 A T-shirt shop carries two competing shirts, one endorsed by Michael Jordan and the other by Barry Bonds. The owner of the store can obtain both types at a cost of \$2 per shirt and estimates that if Jordan shirts are sold for x dollars apiece and Bonds shirts for y dollars apiece, consumers will buy approximately $40 - 50x + 40y$ Jordan shirts and $20 + 60x - 70y$ Bonds shirts each day. How should the owner price the shirts in order to generate the largest possible profit?

(Answer: $x = \$2.70, y = \$2.50$)

ហាងមួយលក់អាវយឺតពីរប្រភេទ ។ ប្រភេទទី១គឺអាវ Michael Jordan មួយប្រភេទទៀតគឺអាវ Barry Bonds ។ ម្ចាស់ហាងទិញចូលអាវយឺតទាំងពីរប្រភេទនេះដោយថ្លៃដើម 2ដុល្លាក្នុង១អាវ ។ គាត់រំពឹងថាបើលក់អាវ Jordan ថ្លៃ xដុល្លាក្នុង១អាវ និងអាវBondsលក់ថ្លៃ yដុល្លាក្នុង១អាវ នោះក្នុងមួយថ្ងៃៗអតិថិជននឹងទិញអាវJordanអស់ប្រហែល $40 - 50x + 40y$ អាវ និងទិញអាវBondsអស់ប្រហែលជា

$20+60x-70y$ អាវ ។ តើម្ចាស់ហាងគួរកំណត់ថ្លៃលក់នៃអាវប្រភេទនិមួយៗប៉ុន្មាន ដើម្បីឱ្យបានប្រាក់ចំណេញខ្ពស់បំផុត?

22 The telephone company is planning to introduce two new types of executive communications systems that it hopes to sell to its largest commercial customers. It is estimated that if the first type of system is priced at x hundred dollars per system and the second type at y hundred dollars per system, approximately $40-8x+5y$ consumers will buy the first type and $50+9x-7y$ will buy the second type. If the cost of manufacturing the first type is \$1,000 per system and the cost of manufacturing the second type is \$3,000 per system, how should the telephone company price the systems to generate the largest possible profit? ក្រុមហ៊ុនទូរស័ព្ទមួយគ្រោងនឹងដាក់បង្ហាញប្រព័ន្ធប្រតិបត្តិការទូរគមនាគមន៍ថ្មីចំនួនពីរប្រភេទដែលពួកគេគិតថាអាចនឹងលក់ទៅឱ្យអតិថិជនពាណិជ្ជកម្មធំៗបំផុតរបស់ពួកគេ ។ គេរំពឹងថា បើសិនជាប្រព័ន្ធប្រភេទទី១លក់ក្នុងតម្លៃ x រយដុល្លាក្នុង១ប្រព័ន្ធ និងប្រព័ន្ធប្រភេទទី២លក់ក្នុងតម្លៃyរយដុល្លាក្នុង១ប្រព័ន្ធ នោះអតិថិជនចំនួនប្រហែលជា$40-8x+5y$ នឹងទិញប្រព័ន្ធប្រភេទ១ ហើយអតិថិជនចំនួនប្រហែលជា $50+9x-7y$ នាក់ នឹងទិញប្រព័ន្ធប្រភេទទី២ ។ ប្រសិនបើចំណាយថ្លៃដើមលើការផលិតប្រព័ន្ធប្រភេទទី១ស្មើនឹង1000ដុល្លាក្នុងមួយប្រព័ន្ធ និងប្រភេទទី២ស្មើនឹង3000ដុល្លាក្នុងមួយប្រព័ន្ធ នោះតើក្រុមហ៊ុនទូរស័ព្ទគួរកំណត់ថ្លៃលក់ប៉ុន្មានដើម្បីឱ្យបានប្រាក់ចំណេញខ្ពស់បំផុត?

23 A manufacturer is planning to sell a new product at the price of \$150 per unit and estimates that if x thousand dollars is spent on the development and y thousand dollars is spent on promotion, consumers will buy appoimately $\frac{320y}{y+2}+\frac{160x}{x+4}$ units of the product. If manufacturing costs for this product are \$50 per unit, how much should the manufacturer spend on development and how mnuch on promotion to generate the largest possible profit from the sale of this product? (*Hint:* Profit=(number of units)(price per unit−cost per unit)−totla amount spent on development and promotion)ផលិតករម្នាក់គ្រោងលក់ផលិតផលថ្មីមួយរបស់ខ្លួនក្នុងតម្លៃ50ដុល្លាក្នុង១ឯកតា ហើយគាត់រំពឹងថាបើចំណាយទៅលើការផលិតចំនួនxពាន់ដុល្លា និងលើការផ្សព្វផ្សាយចំនួន yពាន់ដុល្លា នោះអតិថិជននឹងទិញចំនួនប្រហែល$\frac{320y}{y+2}+\frac{160x}{x+4}$ ឯកតានៃផលិតផល ។ បើសិនជាចំណាយថ្លៃដើមទៅលើផលិតផលទាំងនេះស្មើនឹង50ដុល្លាក្នុង១ឯកតា តើផលិតករគួរចំណាយលើការផលិតប៉ុន្មាន និងលើការផ្សព្វផ្សាយប៉ុន្មាន ដើម្បីឱ្យប្រាក់ចំណេញពីការលក់ផលិតផលនេះខ្ពស់បំផុត?

(Answer: \$4000 on development and \$6000 on promotion)

24 A dairy produces whole milk and skim milk in quantities x and y gallons, respectively. Suppose that the price of whole milk is $p(x)=100-x$ and that of skim milk is $q(y)=100-y$ and assume that $C(x,y)=x^2+xy+y^2$ is the joint-cost function of the commodities. What should x and y be in order to maximize profit? កន្លែងផលិតគ្រឿងបរិភោគពីទឹកដោះគោមួយផលិតទឹកដោះគោ whole milk និងទឹកដោះគោ

skim milk ក្នុងចំនួនxនិងyហ្គាឡុងរៀងគ្នា។ ឧបមាថាថ្លៃទឹកដោះគោ whole milk គឺ $p(x)=100-x$ និងទឹកដោះគោskim milk គឺ $q(y)=100-y$ ហើយសន្មតថាអនុគមន៍ចំណាយរួមគឺ$C(x,y)=x^2+xy+y^2$ ។ តើ xនិងyគួរមានតម្លៃប៉ុន្មានដើម្បីឱ្យប្រាក់ចំណេញមានកំរិតអតិបរមា?

25 A dairy produces whole milk and skim milk in quantities x and y gallons, respectively. Supose that the price of whole milk is $p(x)=20-5x$ and that of skim milk is $q(y)=4-2y$ and assume that $C(x,y)=2xy+4$ is the joint-cost function of the commodities. What should x and y be in order to maximize profit? (Answer: $x=2, y=0$).

26 Find the maximum value of the function $f(x,y)=xy$ subject to the constraint $x+y=1$. ចូររកអតិបរមារបស់អនុគមន៍ $f(x,y)=xy$ ក្រោមលក្ខខណ្ឌព្រំដែន $x+y=1$ ។

27 Find the minimum value of the function $f(x,y)=x^2+y^2$ subject to the constraint $xy=1$ ។ រកអប្បបរមារបស់អនុគន៍ $f(x,y)=x^2+y^2$ ក្រោមលក្ខខណ្ឌព្រំដែន$xy=1$ ។

28 Find the minimum value of the function $f(x,y)=x^2+2y^2-xy$ subject to the constraint $2x+y=22$.

29 Find the minimum value of the function $f(x,y)=x^2-y^2$ subject to the constraint $x^2+y^2=4$.

30 Find the maximum and minimum values of the function $f(x,y)=8x^2-24xy+y^2$ subject to the constraint $x^2+y^2=1$.

31 A manufacturer has \$8,000 to spend on the development and promotion of a new product. It is estimated that if x thousand dollars is spent on development and y thousand is spent on promotion, sales will be approximately $f(x,y)=50x^{1/2}y^{3/2}$ units. How much money should the manuacturer allocate to development and how much to promotion to maximize sales? ផលិតករម្នាក់មាន8000ពាន់ដុល្លាសម្រាប់ចំណាយទៅលើការផលិត និងផ្សព្វផ្សាយផលិតផលថ្មីមួយ។ គេរំពឹងទុកថាបើបើចំណាយ xពាន់ដុល្លាទៅលើការផលិត និងyពាន់ដុល្លាទៅលើការផ្សព្វផ្សាយ នោះគេលក់ដាច់ប្រហែល $f(x,y)=50x^{1/2}y^{3/2}$ ឯកតា។ តើផលិតករគួរបែងចែកទៅលើការផលិតប៉ុន្មាន និងទៅលើការផ្សព្វផ្សាយប៉ុន្មានដើម្បីធ្វើឱ្យការលក់ឡើងដល់កំរិតអតិបរមា?

32 If x thousand dollars is spent on labor and y thousand dollars is spent on equipment, the output at a certain factory will be $Q(x,y)=60x^{1/3}y^{2/3}$ units. If \$120,000 is available, how should this be allocated between labor and equipment to generate the largest possible output? បើគេចំណាយxពាន់ដុល្លាទៅលើកំលាំងពលកម្ម និងyពាន់ដុល្លាទៅលើសម្ភារៈ នោះចំនួនផលិតផលចេញពីរោងចក្រមួយកំណត់ដោយ$Q(x,y)=60x^{1/3}y^{2/3}$ ឯកតា។ បើគេមាន120000ដុល្លា នោះតើគេគួរបែងចែកយ៉ាងដូចម្តេចដើម្បីឱ្យចំនួនផលិតផលមានកំរិតអតិបរមា?

33 A manufacturer is planning to sell a new product at the price of \$150 per unit and estimates that if x thousand dollars is spent on development and y thousand dollars is spent on promotion, approximately $\frac{320y}{y+2}+\frac{160x}{x+4}$ units of the product will be sold. The cost of manufacturing the product is \$50 per unit. If the manufacturer has a total of \$8,000 to spend on development and promotion, how should this money be allocated to generate the largest possible profit? (*Hint:* Profit=(number of units)(price per unit−cost per unit)−total amount spent on development and promotion.)ផលិតករម្នាក់គ្រោងលក់ផលិតផលថ្មីក្នុងតម្លៃ150ដុល្លាក្នុង១ឯកតា ហើយរំពឹងទុកថាបើគេចំណាយxពាន់ដុល្លាទៅលើការផលិត និងyពាន់ដុល្លាទៅលើការផ្សព្វផ្សាយ នោះផលិតផលប្រហែលជា$\frac{320y}{y+2}+\frac{160x}{x+4}$ នឹងត្រូវលក់ដាច់ ។ ថ្លៃដើមលើការផលិតគឺ50ដុល្លា ។ បើផលិតករមាន 8000ដុល្លាសម្រាប់ចំណាយលើការផលិត និងផ្សព្វផ្សាយ នោះតើគេគួរបែងចែកដូចម្តេចដើម្បីឱ្យប្រាក់ចំណេញអតិបរមា? (**ណែនាំ:** ប្រាក់ចំណេញ=(ចំនួនឯកតា) (ថ្លៃលក់ក្នុង១ឯកតា-ថ្លៃចំណាយក្នុង១ឯកតា) - ចំណាយសរុបលើការផលិតនិងផ្សព្វផ្សាយ)) ។

34 If x thousand dollars is spent on labor and y thousand dollars is spent on equipment, the output at a certain factory will be $Q(x,y)=60x^{1/3}y^{2/3}$ units. If \$120,000 is available, how should this be allocated between labor and equipment to generate the largest possible output? Use Lagrange multiplier λ to estimate the change in the maximum output of the factory that will result if the money available for labor and equipment is increased by \$1,000. បើគេចំណាយxពាន់ដុល្លា ទៅលើកំលាំងពលកម្ម និងy ពាន់ដុល្លាទៅលើសម្ភារៈ នោះទិន្នផលចេញពីរោងចក្រមួយកំណត់ដោយអនុគមន៍ $Q(x,y)=60x^{1/3}y^{2/3}$ ឯកតា ។ បើមាន120000ដុល្លាសម្រាប់ចំណាយ នោះតើគេគួរចំណាយលើកំលាំងពលកម្មប៉ុន្មាន និងលើសម្ភារៈប៉ុន្មានដើម្បីឱ្យបានទិន្នផលខ្ពស់បំផុត? ប្រើមេគុណឡាក្រង់ λ ដើម្បីប៉ាន់ស្មាននូវទំហំប្រែប្រួលរបស់កំរិតអតិបរមានៃទិន្នផលរបស់រោងចក្របើគេបង្កើនចំណាយចំនួន 1000ដុល្លា ។

35 A consumer has \$280 to spend on two commodities, the first of which costs \$2 per unit and the second \$5 per unit. Suppose that the utility derived by the consumer from x units of the first commodity and y units of the second commodity is $U(x,y)=100x^{0.25}y^{0.75}$. How many units of each commodity should the consumer buy to maximize utility? អតិថិជនម្នាក់បានចំណាយ 280 ដុល្លាទៅលើរបស់ ប្រើប្រាស់ពីរប្រភេទ ។ ប្រភេទទី១ ថ្លៃ2ដុល្លាក្នុង១ឯកតា និងប្រភេទទី២ថ្លៃ5ដុល្លាក្នុង១ឯកតា ។ ឧបមាថា អត្ថប្រយោជន៍ដែលបានមកពីរបស់ប្រើប្រាស់ប្រភេទទី១ចំនួនxឯកតា និងពីរបស់ប្រើប្រាស់ប្រភេទទី២ចំនួនy ឯកតាកំណត់ដោយអនុគមន៍$U(x,y)=100x^{0.25}y^{0.75}$ ។ តើអតិថិជនគួរទិញរបស់ប្រើប្រាស់ប្រភេទនិមួយៗប៉ុន្មានឯកតាធ្វើឱ្យអនុគមន៍អត្ថប្រយោជន៍ដែលបានមក មានកំរិតអតិបរមា?

36 A consumer has k dollars to spend on two commodities, the first of which costs a dollars per unit and the second b dollars per unit. Suppose that the utility derived by the consumer from x units of the first commodity and y units of the second

commodity is given by the Cobb-Douglas utility function $U(x,y)=x^{\alpha}y^{\beta}$, where $0<\alpha<1$ and $\alpha+\beta=1$. Show that utility will be maximized when $x=\dfrac{k\alpha}{a}$ and $y=\dfrac{k\beta}{b}$. អតិថិជនម្នាក់មាន k ដុល្លាសម្រាប់ចំណាយទៅលើរបស់ប្រើប្រាស់ពីរប្រភេទ ។ ប្រភេទទី១ ថ្លៃ aដុល្លាក្នុង១ឯកតា និងប្រភេទទី២ថ្លៃ bដុល្លាក្នុង១ឯកតា ។ ឧបមាថា អត្ថប្រយោជន៍ដែលបានមកពីរបស់ប្រើប្រាស់ប្រភេទទី១ចំនួនxឯកតា និងពីរបស់ប្រើប្រាស់ប្រភេទទី២ចំនួនy ឯកតាកំណត់ដោយអនុគមន៍ $U(x,y)=x^{\alpha}y^{\beta}$ ដែល $0<\alpha<1$ and $\alpha+\beta=1$ ។ ចូរបង្ហាញថា អនុគមន៍អត្ថប្រយោជន៍ឈានដល់អតិបរមាបើ $x=\dfrac{k\alpha}{a}$ and $y=\dfrac{k\beta}{b}$ ។

Chapter 4

Linear Programming (LP)

1 System of Linear Inequalities in Two Variables

Example 1

A manufacturer makes two grades of concrete. Each bag of the high-grade concrete contains 10 kg of gravel and 5 kg of cement, while each bag of low-grade concrete contains 12 kg of gravel and 3 kg of cement. There are 1,920kg of gravel and 780 kg of cement currently available. The manufacturer can make a profit of $1.20 on each bag of the high-grade and $1.00 on each bag of the low-grade concrete and wishes to determine how many bags of each grade to make from the available supplies to generate the largest possible profit. Formulate this problem in mathematical terms.

Solution

We summarize the problem by the table as follows

	High-grade (kg)	Low-grade (kg)	Amount available(kg)
Gravel	10	12	1,920
Cement	5	3	780
Profit per bag	1.20	1.00	

If we let *H* denote the number of bags of high-grade concrete produced and *L* denote the number of bags of low-grade cencrete produced, we can represent the manufacturer ‘s profit *P* by the linear function

$$P = 1.20H + 1.00L$$

with the constraints

$$10H + 12L \leq 1,920 \text{ for gravel}$$

$$5H + 3L \leq 780 \text{ for cement}$$

and also, number of bag must be non-negative. Hence $H \geq 0$ and $L \geq 0$. Therefore the problem can be stated mathematically as follows:

Maximize the profit function $P = 1.20H + 1.00L$

Subject to the constraints

$$10H + 12L \leq 1,920$$

$$5H + 3L \leq 780$$

$$H, L \geq 0$$

Note that in this problem, the optimum point we seek lies in the solution set of a system of linear inequalities.

1.1 Graphing a Linear Inequality in Two Variables

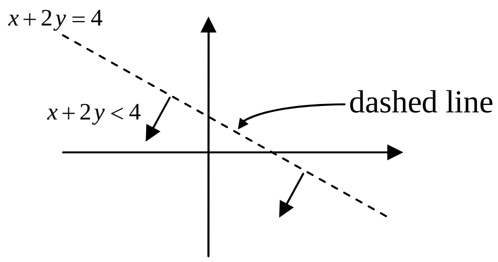
When a plane is divided in half by a line, each side is called a *half-plane*. A vertical line divides the plane into *left and right half-planes*, and any nonveritical line divides the plane into *upper and lower half-planes.*

There are four different kinds of inequalities in two variables that can arise:

$$Ax+By<C \quad Ax+By\le C \quad Ax+By>C \quad Ax+By\ge C$$

where *A*, *B*, and *C* are constants with at least one coefficients *A*, *B* not zero. If the equality invelves < or >, the half-plane is **open**, and we use a dashed line to indicate that the boundary line is not part of the solution set. If the inequalities involves $\le$ or $\ge$, the half-plane is **closed** and we use the solid line to indicate that the boundary is part of the solution set.

Example 2
Graph the inequality $x+2y<4$



Procedure for Graphing a Linear Inequality in Two Variables
To graph an inequality of one of the forms

$$Ax+By<C \quad Ax+By\le C \quad Ax+By>C \quad Ax+By\ge C$$

we proceed as follows:

1. Graph the coresponding equation $Ax+By=C$. Use a dashed line for this line in the < and > cases, and a solid line in the $\le$ and $\ge$ cases.
2. The solution set is the half-plane on one side of the boundary line. To determine which side, choose a test point $P(a,b)$ not one the line and check to see if the coordinates *a* and *b* satisfy the given inequality.

Example 3
Graph each of the following inequalities:
(a) $2x-y\ge 5$ (b) $3x+y>0$ (c) $x<4$

1.2 Solving Systems of Linear Inequalities

To solve a system of linear inequalities such as

$$\begin{aligned} 3x+5y&<2 \\ x-2y&\ge 7 \end{aligned} \quad \text{or} \quad \begin{aligned} x+y&\le 4 \\ 3x-y&<2 \\ x+2y&>3 \end{aligned}$$

we must find the set of all points (x,y) that satisfy all the inequalities in the system simultaneously. In general this solution set will be a region of the plane, which we shall refer to as the *feasibility region* for the system of inequalities. To obtain the feasibility region of a given system of linear inequalities, we first gragph the individual inequalities in the system on the same set of coordinate axes and then take the intersection of these individual graphs.

Example 4

Graph the feasibility region for the following system of inequalities

$$3x - y \le 2$$
$$x + y \ge 6$$

Corner Point

A **corner point** of a particular feasibility region *R* is a point in *R* where two of the boudary lines of *R* intersect.

Example 6

Graph the feasibility region for the following system of inequalities:

$$2x - y \ge 5$$
$$3x + y > 0$$
$$x < 4$$

Answer: the corner points $(1,-3)$ $(4,3)$ $(4,-12)$

Example 7

Graph the feasibility region and find the corner points for the following system of inequalities:

$$10H + 12L \le 1,920$$
$$5H + 3L \le 780$$
$$H, L \ge 0$$

Answer: the corner points $(156,0)$ $(120,60)$ $(0,160)$ $(0,0)$

2 Geometric Linear Programming

In general, a **linear program** in two variables *x* and *y* consists of a linear function $f = ax + by$ called the **objective function** that is to be optimized (maximized or minimized) subject to a system of linear constraint inequalities. The variables *x* and *y* are called **decision variables**, and the solution set of the system of inequality constraints is called the **feasibility region.**

The Corner Point Theorem

If a linear program has an optimal solution (maximum or minimum), it must occur at a corner point of the feasibility region.

Geometric Method for Solving a Linear Program with Two Decision Variables

1. Graph the feasiblity region *R* and find the coordinates of all corner points of *R*.
2. Make a table evaluating the objective function *F* at each corner point.
3. If *R* can be contained in a circle it is **bounded.** In this case, the largest (smallest) value of *F* on *R* is its largest (smallest) value at a corner point.
4. If *R* cannot be contained in a circle, it is **unbounded,** and an optimum soluion may not exist. However, if it does, it must occur at a corner point.

Example 1

From the example 1 in section 1, how many bags of each grade should be made up from available supplies to generate the largest possible profit?

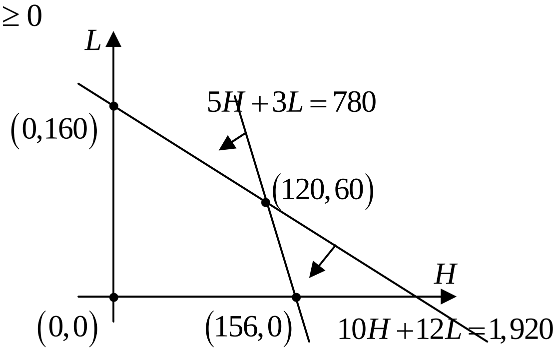
Solution

To maximize

$$P = 1.2H + L$$

Subjecto to

$$10H + 12L \le 1{,}920$$
$$5H + 3L \le 780$$
$$H, L \ge 0$$



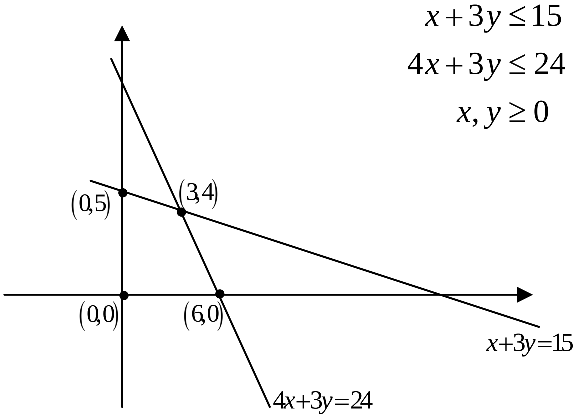
Corner Point	Value of $P = 1.2H + L$
$(156,0)$	187.2
$(120,60)$	204
$(0,160)$	160
$(0,0)$	0

From this table, we see that the largest value of the objective function over the feasibility region occurs at the point $(120,60)$.

Example 2

Find the largest and the smallest values of $F = 3x + 4y$ subject to the following inequalities:

$$x + 3y \le 15$$
$$4x + 3y \le 24$$
$$x, y \ge 0$$

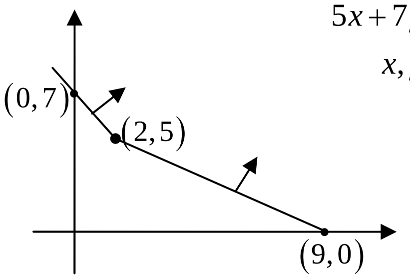


Corner Point	Value of $P = 3x + 4y$
$(0,0)$	0
$(0,5)$	20
$(3,4)$	25
$(6,0)$	18

Thus, the largest value of *F* is 25 at (3, 4), and the smallest value is 0 at the origin.

Example 3
What can be said about the largest and samllest values of the function $F = 4x + 3y$ subject to the following inequalities:

$$x + y \geq 7$$
$$5x + 7y \geq 45$$
$$x, y \geq 0$$



From the figure the feasible region *R* is unbounded (it cannot be contained in a circle) and that its corner points are $(0,7)$ $(2,5)$ $(9,0)$. This suggests that there is a smallest value of *F* and *R* but no largest value. Evaluating the corner points of *R*, we obtain the following table:

Corner Point	Value of $F = 4x + 3y$
$(0,7)$	21
$(2,5)$	23
$(9,0)$	36

Thus, the largest value of *F* doesn't exist and the smallest value of *F* subject to the given constraints is 21 which occurs at the point where $x = 0$ and $y = 7$.

Example 4
A farmer prepares feed for livestock by combining two grains. Each unit of the first grain costs 20 cents and contains 2 units of protein and 5 units of iron, while each unit of the second grain costs 30 cents and contains 4 units of protein and 1 unit of iron. Each animal must receive at least 10 units of protein and 16 units of iron each day. How many units of each grain should the farmer feed to each animal to satisfy these nutritional requirements at the smallest possible cost?

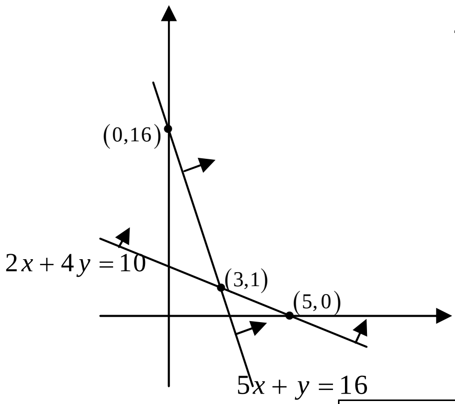
Solution
For convenience, we construct the following table

	Grain I	Grain II	Minimal Requirement
Protein	2	4	10
Iron	5	1	16
Cost	20	30	

If we let
x: number of units of grain I fed daily to each animal
y: number of units of grain II fed daily to each animal

we can represent the farmer's cost by the linear function $C = 20x + 30y$ and the nutritional requirements by the linear inequalities $2x + 4y \geq 10$ and $5x + y \geq 16$. The resulting linear programming problem is to minimize the objective function $C = 20x + 30y$ subject to

$$\begin{aligned} 2x + 4y &\geq 10 \\ 5x + y &\geq 16 \\ x, y &\geq 0 \end{aligned}$$



Corner Point	Value of F=4x+3y
(0, 16)	480
(3, 1)	90
(5, 0)	100

Since 90 is the smallest of these values, the corresponding corner point (3, 1) must be an optimal solution. We conclude that to satify the nutritional requirements at the smallest possible cost, the farmer should feed each animal 3 units of grain I and 1 unit of grain II each day. The minimal cost is then 90 cents per animal.

Exercises

1 In the following problems, graph the feasibility region of the given system of linear inequalities. Find the coordinates of all corner points.ចូរគូសតំបន់ដែលទទួលយកបាននៃចម្លើយរបស់ប្រព័ន្ធវិសមីការខាងក្រោម ព្រមទាំងរកកូអ័រដោនេនៃចំណុចកាច់ជ្រុង ។

1. $\begin{aligned} 2x + 3y &< 0 \\ x + 7y &\geq 0 \end{aligned}$ 2. $\begin{aligned} 4x + 5y &< 9 \\ 2x - y &\geq 1 \\ y &\geq 0 \end{aligned}$ 3. $\begin{aligned} x + y &\leq 7 \\ 2x + 3y &\leq 16 \\ x, y &\geq 0 \end{aligned}$ 4. $\begin{aligned} x + y &\leq 3 \\ x - 2y &\leq 0 \\ x + y &\leq 3 \\ x, y &\geq 0 \end{aligned}$

Formulate the given problem as a linear program

ចូរសរសេររូបមន្តចំណោទកម្មវិធីលីនេអ៊ែរខាងក្រោមនោះ(មិនបាច់ដោះស្រាយ)

2 At a local leather shop, 1 hour of skilled labor and 1 hour of unskilled labor are required to produce a briefcase, while 1 hour of skilled labor and 2 hours of unskilled labor are required to produce a suitcase. The owner of the shop can make a profit of \$15on each briefcase and \$20 on each suitcase. On a particular day, only 7 hours of skilled labor and 11 hours of unskilled labor are available, and the owner wishes to determine how many briefcases and how many suitcases to make that day to generate the largest profit possible. នៅក្នុងហាងផលិតនិងលក់ផលិត

ផលអំពីស្បែកមួយ ដើម្បីផលិតវ៉ាលីសតូចមួយគេត្រូវការកំលាំងពលកម្មជំនាញ១ម៉ោង និងកំលាំងពលកម្មមិនជំនាញក៏១ម៉ោងដែរ។ ចំពោះវ៉ាលីសធំវិញគេត្រូវការកំលាំងពលកម្មជំនាញ១ម៉ោង និងកំលាំងពលកម្មមិនជំនាញចំនួន២ម៉ោង។ ម្ចាស់ហាងបានចំណេញ ១៥ដុល្លាពីវ៉ាលីសតូចនីមួយៗ និង ២០ដុល្លាពីវ៉ាលីសធំនីមួយៗ។ ក្នុងមួយថ្ងៃណាមួយនោះ គេអាចប្រើកំលាំងពលកម្មជំនាញ និងមិនជំនាញបានតែ៧ម៉ោង និង១១ម៉ោងរៀងគ្នាតែប៉ុណ្ណោះ។ តើនៅក្នុងថ្ងៃនោះគេគួរផលិតនិងលក់វ៉ាលីសតូចប៉ុន្មាន និងធំប៉ុន្មានដើម្បីឱ្យបានប្រាក់ចំណេញខ្ពស់បំផុត?

3 A dietician is planning a meal consisting of three foods whose ingredients (per unit) are summarized in the following table:

Ingredients	Food I	Food II	Food III
Units of protein	5	10	15
Units of carbohydrates	2	3	2
Units of iron	3	6	1
Calories	60	140	120

The dietician wishes to determine the number of units of each food needed to create a meal containing at least 30 units of protein, 8 units of carbohydrates, and 10 units of iron, with as few calories as possible.

អ្នកជំនាញខាងការតមអាហារកំពុងនឹងគ្រោងរៀបចំអាហារមួយពេលដែលមានមុខម្ហូបចំនួន៣មុខដែលគ្រឿងផ្សំ(ក្នុង១ឯកតា) មានដូចក្នុងតារាងខាងក្រោមនេះ

គ្រឿងផ្សំ	អាហារ I	អាហារ II	អាហារ III
ចំនួនឯកតានៃប្រូតេអ៊ីន	5	10	15
ចំនួនឯកតាកាបូនអ៊ីដ្រាត	2	3	2
ចំនួនឯកតានៃសារជាតិដែក	3	6	1
កាឡូរី	60	140	120

អ្នកជំនាញខាងការតមអាហារនោះចង់កំណត់ចំនួនឯកតានៃមុខម្ហូបនីមួយៗដើម្បីឱ្យបានអាហារមួយពេលដែលមានយ៉ាងហោចជាតិប្រូតេអ៊ីន 30ឯកតា កាបូនអ៊ីដ្រាត8ឯកតា និងជាតិដែក10ឯកតា និងក៏រិតកាឡូរី តិចបំផុតតាមដែលអាចធ្វើបាន។

Use graphical technique to solve the given linear programming problems

ដោយប្រើវិធីក្រាហ្វិច ចូរដោះស្រាយចំណោទកម្មវិធីលីនេអ៊ែរដូចខាងក្រោម

4 Find the largest and smallest values of $F = 5x + y$ subject to

ចូររកតម្លៃធំបំផុត និងតូចបំផុតរបស់អនុគមន៍ $F = 5x + y$ ក្រោមល័ក្ខខ័ណ្ឌ

$$4x + 3y \le 24$$
$$x + 3y \le 15$$
$$x, y \ge 0$$

5 Find the smallest value of $F = 20x + 80y$ subject to:

$$4x + y \geq 8$$
$$2x + 3y \geq 14$$
$$2x + 5y \geq 18$$

Does Fhave a largest value on this feasibility region?

តើ F មានតម្លៃធំបំផុត នៅលើតំបន់ដែលអាចទទួលយកបាននៃចម្លើយឬទេ ចំពោះលំហាត់ 6-8 ?

6 Maximize and minimize the function $F = 3x + y$ subject to the following constraints

$$3x - 2y \geq 0$$
$$3x + y \leq 18$$
$$y \geq 3$$
$$x, y \geq 0$$

7 Minimize the function $F = 2x + 3y$ subject to the following constraints

$$x + 3y \geq 19$$
$$4x + 5y \geq 34$$
$$3x + 2y \geq 22$$
$$x, y \geq 0$$

8 Minimize the function $F = 3x + y$ subject to the following constraints

$$3x + 5y \leq 38$$
$$x - 3y \leq -6$$
$$2x + y \geq 9$$
$$x, y \geq 0$$

9 At a local leather shop, 1 hour of skilled labor and 1 hour of unskilled labor are required to produce a briefcase, while 1 hour of skilled labor and 2 hours of unskilled labor are required to produce a suitcase. The owner of the shop can make a profit of $15on each briefcase and $20 on each suitcase. On a particular day, only 7 hours of skilled labor and 11 hours of unskilled labor are available, and the owner wishes to determine how many briefcases and how many suitcases to make that day to generate the largest profit possible. នៅក្នុងហាងផលិតនិងលក់ផលិតផលអំពីស្បែកមួយ ដើម្បីផលិតវ៉ាលីសតូចមួយគេត្រូវការកំលាំងពលកម្មជំនាញ១ម៉ោង និងកំលាំងពលកម្មមិនជំនាញក៏១ម៉ោងដែរ។ ចំពោះវ៉ាលីសធំវិញគេត្រូវការកំលាំងពលកម្មជំនាញ១ម៉ោង និងកំលាំងពលកម្មមិនជំនាញចំនួន២ម៉ោង។ ម្ចាស់ហាងបានចំណេញ ១៥ដុល្លាពីវ៉ាលីសតូចនីមួយៗ និង ២0ដុល្លាពីវ៉ាលីសធំនីមួយៗ។ ក្នុងមួយថ្ងៃណាមួយនោះ គេអាចប្រើកំលាំងពលកម្មជំនាញ និងមិនជំនាញបានតែ៧ម៉ោង និង១១ម៉ោងរៀងគ្នាតែប៉ុណ្ណោះ។ តើនៅក្នុងថ្ងៃនោះគេគួរផលិតនិងលក់វ៉ាលីសតូចប៉ុន្មាន និងធំប៉ុន្មានដើម្បីឱ្យបានប្រាក់ចំណេញខ្ពស់បំផុត?

10 Shipments from one wholesaler contain 3 units of item A, 6 units of item B, 4 units of item C, and cost $20. Shipments from a second wholesaler contain 12 units of item A, 3 units of item B, 3 units of item C, and cost $26. If a retailer requires at least 396 units of A, 288 units of B and 255 units of C, how many shipments from each wholesaler should be ordered to minimize total cost?

នៅក្នុងការដឹកជញ្ជូនចេញពីឃ្លាំងមួយរួមមានមុខទំនិញ Aចំនួន 3ឯកតា មុខទំនិញBចំនួន6ឯកតា និងមុខទំនិញCចំនួន4ឯកតា។ សោហ៊ុយក្នុងការដឹកជញ្ជូននេះគឺ 20ដុល្លា។ ការដឹកជញ្ជូនចេញពីឃ្លាំងមួយទៀតរួមមានមុខទំនិញ Aចំនួន12ឯកតា មុខទំនិញBចំនួន 3ឯកតា និងមុខទំនិញCចំនួន3 ឯកតា ហើយចំណាយ ក្នុងការដឹកជញ្ជូនគឺ 26ដុល្លា។ បើអ្នកទទួលលក់បន្តត្រូវការយ៉ាងហោចណាស់មុខទំនិញ Aចំនួន396ឯកតា មុខទំនិញBចំនួន288ឯកតា និងមុខទំនិញCចំនួន255ឯកតា នោះតើគួរដឹកជញ្ជូនពីឃ្លាំងនីមួយៗប៉ុន្មានជើងដើម្បីឱ្យចំណាយក្នុងការដឹកជញ្ជូននេះតិចបំផុត?

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